Rational choice and AGM belief revision

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Abstract

We establish a correspondence between the rationalizability of choice studied in the revealed preference literature and the notion of minimal belief revision captured by the AGM postulates. A choice frame consists of a set of alternatives Ω, a collection $\mathcal{E}$ of subsets of Ω (representing possible choice sets) and a function $f : \mathcal{E} \to 2^{\Omega}$ (representing choices made). A choice frame is *rationalizable* if there exists a total pre-order R on Ω such that, for every $E \in \mathcal{E}$, $f(E)$ coincides with the best elements of E relative to R. We re-interpret choice structures in terms of belief revision. An interpretation is obtained by adding a valuation V that assigns to every atom p the subset of Ω at which p is true. Associated with an interpretation is an initial belief set and a partial belief revision function. A choice frame is *AGM-consistent* if, for every interpretation of it, the associated partial belief revision function can be extended to a full-domain belief revision function that satisfies the AGM postulates. It is shown that a finite choice frame is AGM-consistent if and only if it is rationalizable.

1 Introduction

The dominant theory of belief revision is due to Alchourrón, Gärdenfors and Makinson [1] and is known as the AGM theory. In their approach beliefs are modeled syntactically as sets of formulas and belief revision is construed as an operation that associates with every deductively closed set of formulas K (thought of as the initial beliefs) and formula ϕ (thought of as new information) a new set of formulas $B_K(\phi)$ representing the new beliefs after revising by ϕ.

We establish a correspondence between the AGM theory and the set-theoretic structures studied in rational choice theory (also known as revealed preference theory; see, for example, [22] and [24]). Rational choice theory considers structures $\langle \Omega, \mathcal{E}, f \rangle$ consisting of a set of alternatives Ω, a collection $\mathcal{E}$ of subsets of Ω (representing possible choice sets) and a function f from $\mathcal{E}$ into the set of subsets of Ω, representing choices made. The main objective of rational choice theory is to investigate the conditions under which the function f can be rationalized by a total pre-order R on Ω in the sense that, for every $E \in \mathcal{E}$, $f(E)$ coincides with the best elements of E relative to R.

We re-interpret choice structures in terms of belief revision. The set Ω is now interpreted as a set of states. A model based on (or an interpretation of) a choice structure is obtained by adding to it a valuation V that assigns to every atomic formula p the set of states at which p is true. Truth of an arbitrary formula at a state is then obtained as usual. Given a model

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$\langle \Omega, \mathcal{E}, f, V \rangle$ we define the *initial beliefs* as the set of formulas ϕ such that $f(\Omega)$ is a subset of the truth set of ϕ, denoted by $||\phi||$. Hence $f(\Omega)$ is interpreted as the set of states that are initially considered possible. We then interpret the collection of events (sets of states) $\mathcal{E}$ as a set of possible items of information. If ϕ is a formula such that $\|\phi\| \in \mathcal{E}$, we define the *revised beliefs upon learning that* ϕ as the set of formulas ψ such that $f(||\phi||) \subseteq ||\psi||$. Thus the event $f(||\phi||)$ is interpreted as the set of states that are considered possible after learning that ϕ is the case. Hence associated with every model is a partial belief revision function (partial because, in general, it is not the case that, for every formula ϕ, $\|\phi\| \in \mathcal{E}$, that is, not every piece of information is potentially available or contemplated). We say that a choice frame $\langle \Omega, \mathcal{E}, f \rangle$ is *AGM-consistent* if, for every model based on it, the associated partial belief revision function can be extended to a full-domain belief revision function that satisfies the AGM postulates. We show that, when the set of states is finite, the properties of AGM-consistency and rationalizability are equivalent.

In the next section we review the notion of belief function and the AGM postulates. In Section 3 we develop the correspondence between AGM belief revision and rational choice. Section 4 contains a brief discussion of related literature and concluding remarks.

2 Belief revision functions

Let Φ be the set of formulas of a propositional language based on a countable set A of atomic formulas.[1] Given a subset $K \subseteq \Phi$, its PL-deductive closure $[K]^{PL}$ (where 'PL' stands for Propositional Logic) is defined as follows: $\psi \in [K]^{PL}$ if and only if there exist $\phi_1, ..., \phi_n \in K$ (with $n \geq 0$) such that $(\phi_1 \wedge ... \wedge \phi_n) \rightarrow \psi$ is a tautology (that is, a theorem of Propositional Logic). A set $K \subseteq \Phi$ is *consistent* if $[K]^{PL} \neq \Phi$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to $[K]^{PL}$). A set $K \subseteq \Phi$ is *deductively closed* if $K = [K]^{PL}$. A *belief set* is a set $K \subseteq \Phi$ which is deductively closed.

Let K be a consistent belief set representing the agent's initial beliefs and let $\Psi \subseteq \Phi$ be a set of formulas representing possible items of information. A *belief revision function based on* K is a function $B_K : \Psi \rightarrow 2^\Phi$ (where 2^Φ denotes the set of subsets of Φ) that associates with every formula $\phi \in \Psi$ (thought of as new information) a set $B_K(\phi) \subseteq \Phi$ (thought of as the revised beliefs).[2] If $\Psi \neq \Phi$ then B_K is called a *partial* belief revision function, while if $\Psi = \Phi$ then B_K is called a *full* belief revision function.

Definition 1 *Let $B_K : \Psi \rightarrow 2^\Phi$ be a (partial) belief revision function and $B_K^* : \Phi \rightarrow 2^\Phi$ a full belief revision function. We say that B_K^* is an* extension *of B_K if, for every $\phi \in \Psi$, $B_K^*(\phi) = B_K(\phi)$.*

A *full* belief revision function is called an *AGM function* if it satisfies the following properties, known as the AGM postulates: $\forall \phi, \psi \in \Phi$,

(AGM1)	$B_K(\phi) = [B_K(\phi)]^{PL}$
(AGM2)	$\phi \in B_K(\phi)$
(AGM3)	$B_K(\phi) \subseteq [K \cup \{\phi\}]^{PL}$
(AGM4)	if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq B_K(\phi)$
(AGM5)	$B_K(\phi) = \Phi$ if and only if ϕ is a contradiction
(AGM6)	if $\phi \leftrightarrow \psi$ is a tautology then $B_K(\phi) = B_K(\psi)$
(AGM7)	$B_K(\phi \wedge \psi) \subseteq [B_K(\phi) \cup \{\psi\}]^{PL}$
(AGM8)	if $\neg\psi \notin B_K(\phi)$, then $[B_K(\phi) \cup \{\psi\}]^{PL} \subseteq B_K(\phi \wedge \psi)$.

[1] Thus Φ is defined recursively as follows: if $p \in A$ then $p \in \Phi$ and if $\phi, \psi \in \Phi$ then $\neg\phi \in \Phi$ and $(\phi \vee \psi) \in \Phi$.

[2] In the literature it is common to use the notation K_ϕ^* or $K * \phi$ instead of $B_K(\phi)$, but for our purposes the latter notation is clearer.

AGM1 requires the revised belief set to be deductively closed. AGM2 requires that the information be believed. AGM3 says that beliefs should be revised minimally, in the sense that no new formula should be added unless it can be deduced from the information received and the initial beliefs.[3] AGM4 says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs. AGM5 requires the revised beliefs to be consistent, unless the information ϕ is a contradiction (that is, $\neg\phi$ is a tautology). AGM6 requires that if ϕ is propositionally equivalent to ψ then the result of revising by ϕ be identical to the result of revising by ψ. AGM7 and AGM8 are a generalization of AGM3 and AGM4 that

> "applies to *iterated* changes of belief. The idea is that if $B_K(\phi)$ is a revision of K [prompted by ϕ] and $B_K(\phi)$ is to be changed by adding further sentences, such a change should be made by using expansions of $B_K(\phi)$ whenever possible. More generally, the minimal change of K to include both ϕ and ψ (that is, $B_K(\phi \wedge \psi)$) ought to be the same as the expansion of $B_K(\phi)$ by ψ, so long as ψ does not contradict the beliefs in $B_K(\phi)$" (Gärdenfors [12], p. 55; notation changed to match ours).[4]

We now turn to a semantics for belief revision, using structures that are known in rational choice theory as choice functions. We shall call them *choice frames*.

3 Choice frames and AGM belief revision

Definition 2 *A* choice frame *is a triple* $\langle \Omega, \mathcal{E}, f \rangle$ *where*

Ω *is a non-empty set of* states *(or possible worlds); subsets of* Ω *are called* events.

$\mathcal{E} \subseteq 2^\Omega$ *is a collection of events (*2^Ω *denotes the set of subsets of* Ω*) such that* $\varnothing \notin \mathcal{E}$ *and* $\Omega \in \mathcal{E}$.

$f : \mathcal{E} \to 2^\Omega$ *is a function that associates with every event* $E \in \mathcal{E}$ *an event* $f(E)$ *satisfying the following properties: (1)* $f(E) \subseteq E$ *and (2)* $f(E) \neq \varnothing$.

In rational choice theory a set $E \in \mathcal{E}$ is interpreted as a set of available alternatives and $f(E)$ is interpreted as the subset of E which consists of the chosen alternatives (see, for example, [22] and [24]). In our case, we think of the elements of $\mathcal{E}$ as possible items of information and the interpretation of $f(E)$ is that, if informed that event E has occurred, the agent considers as possible all and only the states in $f(E)$. The set $f(\Omega)$ is interpreted as the states that are initially considered possible.[5]

In order to interpret a choice frame $\langle \Omega, \mathcal{E}, f \rangle$ in terms of belief revision we need to add a *valuation* $V : A \to 2^\Omega$ that associates with every atomic formula $p \in A$ the set of states at which p is true. The quadruple $\langle \Omega, \mathcal{E}, f, V \rangle$ is called a *model* (or an *interpretation*) *of* $\langle \Omega, \mathcal{E}, f \rangle$. Given a model $\mathcal{M} = \langle \Omega, \mathcal{E}, f, V \rangle$, truth of an arbitrary formula at a state is defined recursively as follows ($\omega \models_{\mathcal{M}} \phi$ means that formula ϕ is true at state ω in model $\mathcal{M}$): (1) for $p \in A$, $\omega \models_{\mathcal{M}} p$ if and only if $\omega \in V(p)$, (2) $\omega \models_{\mathcal{M}} \neg\phi$ if and only if $\omega \not\models_{\mathcal{M}} \phi$ and (3)

[3] For every formula ψ, $\psi \in [K \cup \{\phi\}]^{PL}$ if and only if $(\phi \to \psi) \in K$ (since, by hypothesis, $K = [K]^{PL}$).

[4] The expansion of $B_K(\phi)$ by ψ is $[B_K(\phi) \cup \{\psi\}]^{PL}$. Note, again, that, for every formula χ, $\chi \in [B_K(\phi) \cup \{\psi\}]^{PL}$ if and only if $(\psi \to \chi) \in B_K(\phi)$ (since, by AGM1, $B_K(\phi) = [B_K(\phi)]^{PL}$).

[5] Notice that, in general, $\mathcal{E}$ may be a "small" subset of 2^Ω. In the revealed preference approach, this is because one might only have a limited number of observations concerning the choices made by an individual (given menu sets $E_1, ..., E_n$ the agent was observed choosing $f(E_1), ..., f(E_n)$, respectively). In the belief revision interpretation, an introspective agent (e.g. a doctor) might consider how she would change her beliefs if she received various pieces of information (e.g. laboratory results), but might be able, or willing, to consider only a limited number of possible items of information.

$\omega \models_{\mathcal{M}} (\phi \vee \psi)$ if and only if either $\omega \models_{\mathcal{M}} \phi$ or $\omega \models_{\mathcal{M}} \psi$ (or both). The truth set of formula ϕ in model $\mathcal{M}$ is denoted by $\|\phi\|_{\mathcal{M}}$. Thus $\|\phi\|_{\mathcal{M}} = \{\omega \in \Omega : \omega \models_{\mathcal{M}} \phi\}$.[6]

Given a model $\mathcal{M} = \langle \Omega, \mathcal{E}, f, V \rangle$ we say that

- the agent *initially believes that* ψ if and only if $f(\Omega) \subseteq \|\psi\|_{\mathcal{M}}$,
- the agent *believes that* ψ *upon learning that* ϕ if and only if (1) $\|\phi\|_{\mathcal{M}} \in \mathcal{E}$ and (2) $f(\|\phi\|_{\mathcal{M}}) \subseteq \|\psi\|_{\mathcal{M}}$.

Accordingly, we can associate with every model a (partial) belief revision function as follows. Let

$$\begin{aligned}
&K_{\mathcal{M}} = \{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|_{\mathcal{M}}\},^{7} \\
&\Psi_{\mathcal{M}} = \{\phi \in \Phi : \|\phi\|_{\mathcal{M}} \in \mathcal{E}\}, \\
&B_{K_{\mathcal{M}}} : \Psi_{\mathcal{M}} \to 2^{\Phi} \text{ given by } B_{K_{\mathcal{M}}}(\phi) = \{\psi \in \Phi : f(\|\phi\|_{\mathcal{M}}) \subseteq \|\psi\|_{\mathcal{M}}\}.
\end{aligned} \tag{1}$$

We address the following question: what properties must a choice frame satisfy in order for it to be the case that the (typically partial) belief revision function associated with an *arbitrary* interpretation (or model) of it can be extended to a full AGM belief revision function? This is the motivation for the following definition.

Definition 3 *A choice frame $\langle \Omega, \mathcal{E}, f \rangle$ is* AGM-consistent *if, for every model $\mathcal{M} = \langle \Omega, \mathcal{E}, f, V \rangle$ based on it, the (partial) belief revision function $B_{K_{\mathcal{M}}}$ associated with $\mathcal{M}$ (see (1)) can be extended (see Definition 1) to a full belief revision function that satisfies the AGM postulates.*

We want to find necessary and sufficient conditions for a choice frame to be AGM-consistent.

Remark 4 *It is shown in the Appendix (Lemma 17) that a necessary condition for AGM-consistency is the following, which is known in the rational choice literature as* Arrow's Axiom *(see [24], p. 25):*

$$\forall E, F \in \mathcal{E}, \text{ if } E \subseteq F \text{ and } E \cap f(F) \neq \varnothing \text{ then } f(E) = E \cap f(F). \tag{2}$$

Arrow's Axiom, however, is not sufficient for AGM-consistency, as the following example shows:[8]

$$\begin{aligned}
&\Omega = \{\alpha, \beta, \gamma, \delta, \varepsilon\}, \quad \mathcal{E} = \{\Omega, \{\alpha, \beta, \gamma\}, \{\beta, \gamma, \delta\}\} \\
&f(\Omega) = \{\varepsilon\}, \quad f(\{\alpha, \beta, \gamma\}) = \{\gamma\} \text{ and } f(\{\beta, \gamma, \delta\}) = \{\beta, \gamma\}.
\end{aligned}$$

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[6] A valuation V (and corresponding model $\mathcal{M}$) associates with every state $\omega \in \Omega$ a maximally consistent set of formulas $m(\omega) = \{\phi \in \Phi : \omega \models_{\mathcal{M}} \phi\}$. Let $\mathbb{M}$ denote the set of maximally consistent sets of formulas. Then a valuation is equivalent to a choice of a label function from Ω to $\mathbb{M}$ (see [19] and [21]).

[7] It is straightforward to show that, for every model $\mathcal{M}$, $K_{\mathcal{M}}$ is a consistent and deductively closed set (a proof can be found in [8]).

[8] Another well-known condition, which is necessary but not sufficient for AGM-consistency, is the Weak Axiom of Revealed Preference ($WARP$):

$$\text{if } E, F \in \mathcal{E},\ x, y \in E \cap F,\ x \in f(E) \text{ and } y \in f(F) \text{ then } x \in f(F)$$

(equivalently, if $E, F \in \mathcal{E}$, $E \cap f(F) \neq \varnothing$ and $f(E) \cap F \neq \varnothing$ then $E \cap f(F) = f(E) \cap F$).

WARP is stronger than Arrow's Axiom as can be seen in the example of Figure 1, which satisfies Arrow's Axiom but not $WARP$ (take $E = \{\alpha, \beta, \gamma\}$ and $F = \{\beta, \gamma, \delta\}$). To see that $WARP$ is not sufficient for AGM-consistency consider the following frame: $\Omega = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{E} = \{\Omega, \{\alpha, \beta\}, \{\beta, \gamma\}, \{a, \gamma\}\}$, $f(\Omega) = \{\delta\}$, $f(\{\alpha, \beta\}) = \{\alpha\}$, $f(\{\beta, \gamma\}) = \{\beta\}$ and $f(\{\alpha, \gamma\}) = \{\gamma\}$. This frame satisfies $WARP$ vacuously, but is not rationalizable (see Definition 5 and Proposition 6) and thus, by Proposition 8, is not AGM-consistent.

This choice frame is illustrated in Figure 1, where the elements of $\mathcal{E}$ are shown as rectangles and the values of the function f are shown as ovals inside the rectangles. This choice frame satisfies Arrow's Axiom trivially (for $E, F \in \mathcal{E}$, $E \subseteq F$ if and only if $F = \Omega$ and, for every $E \in \mathcal{E}\backslash\{\Omega\}$, $f(\Omega) \cap E = \varnothing$). Consider the model based on this frame where, for some atoms p, q, r and s, $||p|| = \{\alpha, \beta, \gamma\}$, $||q|| = \{\beta, \gamma, \delta\}$, $||r|| = \{\beta, \gamma\}$ and $||s|| = \{\gamma\}$. The initial beliefs are given by the consistent and deductively closed set $K = \{\phi \in \Phi : \varepsilon \models \phi\}$. For every formula ϕ such that $\|\phi\| \in \mathcal{E}$, let $B_K(\phi) = \{\psi \in \Phi : f(\|\phi\|) \subseteq \|\psi\|\}$ be the revised beliefs after receiving information ϕ. It is straightforward to check that $\{(q \wedge r), s\} \subseteq B_K(p)$ and $(p \wedge r) \in B_K(q)$ while $s \notin B_K(q)$ (since $f(\|q\|) = \{\beta, \gamma\} \nsubseteq \{\gamma\} = \|s\|$). Since $s \in B_K(p)$ and $s \notin B_K(q)$,

$$B_K(p) \neq B_K(q). \tag{3}$$

Suppose that $B_K^* : \Phi \to 2^\Phi$ is an AGM function that extends B_K. Since $(q \wedge r) \in B_K(p) = B_K^*(p)$ and $B_K^*(p)$ is consistent, $\neg(q \wedge r) \notin B_K^*(p)$. It follows from AGM7 and AGM8 that $B_K^*(p \wedge (q \wedge r)) = [B_K^*(p) \cup \{(q \wedge r)\}]^{PL} = [B_K^*(p)]^{PL} = B_K^*(p)$. Similarly, since $(p \wedge r) \in B_K(q) = B_K^*(q)$, by AGM7 and AGM8 $B_K^*(q \wedge (p \wedge r)) = B_K^*(q)$. Furthermore, since $(p \wedge (q \wedge r)) \leftrightarrow (q \wedge (p \wedge r))$ is a tautology, it follows from AGM6 that $B_K^*(p \wedge (q \wedge r)) = B_K^*(q \wedge (p \wedge r))$. Hence $B_K^*(p) = B_K^*(q)$. Since B_K^* is an extension of B_K, $B_K^*(p) = B_K(p)$ and $B_K^*(q) = B_K(q)$, yielding a contradiction with (3).

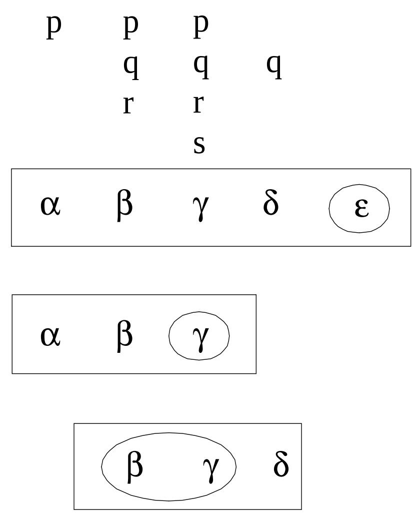

Figure 1
A choice frame that satisfies Arrow's Axiom and a model based on it.

In rational choice theory a choice frame $\langle \Omega, \mathcal{E}, f \rangle$ is said to be *rationalizable* if there exists a total pre-order[9] R on Ω such that, for every $E \in \mathcal{E}$, $f(E)$ is the set of best elements of E relative to R (see Definition 5 below). In that context, the relation R is interpreted as a preference relation ($\omega R \omega'$ if and only if ω is considered to be at least as good as ω'). In our case R can be interpreted as a plausibility relation: $\omega R \omega'$ if and only if state ω is considered to be at least as plausible as state ω'. Given this interpretation, if the frame is rationalizable then, after receiving information E, the agent considers as possible (according to his revised beliefs) all and only the states that are most plausible among the ones in E.

[9]A binary relation $R \subseteq \Omega \times \Omega$ is a total pre-order if it satisfies the following properties:
Completeness: $\forall \omega, \omega' \in \Omega$, either $\omega R \omega'$ or $\omega' R \omega$ (or both),
Transitivity: $\forall \omega, \omega', \omega'' \in \Omega$, if $\omega R \omega'$ and $\omega' R \omega''$ then $\omega R \omega''$.
Note that completeness implies reflexivity ($\forall \omega \in \Omega$, $\omega R \omega$).

Definition 5 *A choice frame $\langle \Omega, \mathcal{E}, f \rangle$ is* rationalizable *if there exists a total pre-order R on Ω such that, for every $E \in \mathcal{E}$,*

$$f(E) = best_R\ E \overset{def}{=} \{\omega \in E : \omega R \omega', \forall \omega' \in E\}.$$ [10]

The following proposition, due to Hansson ([16], Theorem 7, p. 455) gives a necessary and sufficient condition for a choice frame to be rationalizable.

Proposition 6 *A choice frame $\langle \Omega, \mathcal{E}, f \rangle$ is rationalizable if and only if it satisfies the following property: for every sequence $\langle E_1, ..., E_n, E_{n+1} \rangle$ in $\mathcal{E}$ with $E_{n+1} = E_1$, if $E_k \cap f(E_{k+1}) \neq \varnothing$, $\forall k = 1, ..., n$, then $E_k \cap f(E_{k+1}) = f(E_k) \cap E_{k+1}$, $\forall k = 1, ..., n$.*

For instance, in the example illustrated in Figure 1, letting $n = 2$, $E_1 = \{\alpha, \beta, \gamma\}$, $E_2 = \{\beta, \gamma, \delta\}$ and $E_3 = E_1$ we have that $E_1 \cap f(E_2) = \{\beta, \gamma\} \neq \varnothing$ and $E_2 \cap f(E_1) = \{\gamma\} \neq \varnothing$ and thus $E_1 \cap f(E_2) \neq E_2 \cap f(E_1)$, so that, by Proposition 6, the frame is not rationalizable.

The following propositions are proved in the Appendix. The first states that, when Ω is finite, rationalizability implies AGM-consistency and the second that, when Ω is countable, AGM-consistency implies rationalizability.

Proposition 7 *Let $\langle \Omega, \mathcal{E}, f \rangle$ be a choice frame where Ω is finite. If $\langle \Omega, \mathcal{E}, f \rangle$ is rationalizable then it is AGM-consistent.*

Proposition 8 *Let $\langle \Omega, \mathcal{E}, f \rangle$ be a choice frame where Ω is a (possibly infinite) countable set. If $\langle \Omega, \mathcal{E}, f \rangle$ is AGM-consistent then it is rationalizable.*

Putting together Propositions 7 and 8 we get that, when the set of states is finite, the two properties of AGM-consistency and rationalizability are equivalent.

Corollary 9 *Let $\langle \Omega, \mathcal{E}, f \rangle$ be a choice frame where Ω is finite. Then the following are equivalent:*
(a) $\langle \Omega, \mathcal{E}, f \rangle$ is AGM-consistent,
(b) $\langle \Omega, \mathcal{E}, f \rangle$ is rationalizable.

The restriction to finite frames in Proposition 7 can be dropped if the frame $\langle \Omega, \mathcal{E}, f \rangle$ is rationalizable by a total pre-order R that satisfies the property that every non-empty subset of Ω has a best element.

Definition 10 *A choice frame $\langle \Omega, \mathcal{E}, f \rangle$ is* strongly rationalizable *if it is rationalizable by a total pre-order $R \subseteq \Omega \times \Omega$ such that, for every non-empty $E \subseteq \Omega$, $best_R\ E \neq \varnothing$.*[11]

A rationalizable choice frame $\langle \Omega, \mathcal{E}, f \rangle$ where Ω is finite is strongly rationalizable. Thus Proposition 7 is a corollary of the following result (a proof can be found in [8]).

Proposition 11 *Let $\langle \Omega, \mathcal{E}, f \rangle$ be a strongly rationalizable choice frame. Then $\langle \Omega, \mathcal{E}, f \rangle$ is AGM-consistent.*

[10] In the rational choice literature the preference relation is usually denoted by $\succeq$ and the set $\{\omega \in E : \omega \succeq \omega', \forall \omega' \in E\}$ is referred to as the set of *maximal* elements of E. In the artificial intelligence literature, the preference or plausibility relation is usually denoted by $\leq$ and the set $\{\omega \in E : \omega \leq \omega', \forall \omega' \in E\}$ is referred to as the set of *minimal* elements of E. In order to avoid confusion, we denote the relation by R and refer to the *best* elements of a set.

[11] A rationalizable frame may fail to be strongly rationalizable. For example, let $\mathbb{N}$ denote the set of natural numbers and let $\Omega = \mathbb{N} \cup \{\infty\}$. Let $\mathcal{G}$ be the set of *finite* subsets of $\mathbb{N}$ and $\mathcal{E} = \mathcal{G} \cup \{\Omega\}$. Finally, let $f(\Omega) = \{\infty\}$ and, for every $E \in \mathcal{G}$, let $f(E)$ be the largest number in E. Then the choice frame $\langle \Omega, \mathcal{E}, f \rangle$ so defined is rationalizable by the total pre-order $\geq$ (with the convention that $\infty > n$ for every $n \in \mathbb{N}$). However, it is not strongly rationalizable. Suppose it were strongly rationalizable in terms of a total preorder R on Ω. Then $\varnothing \neq best_R\ \mathbb{N} \subseteq \mathbb{N}$. Fix an arbitrary $n \in best_R\ \mathbb{N}$ and let $E = \{n, n+1\}$. By hypothesis, $f(E) = best_R\ E$. Since $E \subseteq \mathbb{N}$ and $E \cap best_R\ \mathbb{N} \neq \varnothing$,

$$best_R\ E = E \cap best_R\ \mathbb{N} = \begin{cases} \{n\} & \text{if } n+1 \notin best_R\ \mathbb{N} \\ \{n, n+1\} & \text{if } n+1 \in best_R\ \mathbb{N}. \end{cases}$$

But $f(E) = \{n+1\}$, yielding a contradiction.

4 Related literature and conclusion

Choice frames provide a semantics for AGM belief revision which can be considered an extension of Grove's [15] system of spheres semantics to the class of *partial* belief revision functions. In the Appendix we review the notion of system of spheres and prove Propositions 7 and (a weaker version of) 8 using Grove's characterization of AGM functions.[12]

Choice frames offer a Kripke-style (Kripke [20]) semantics for belief revision. Recently there have been several attempts to model belief revision along the lines pioneered by Hintikka [17] for static beliefs, namely using a modal logic framework that, on the semantic side, relies on Kripke-style structures. Important work in this new area was done by Segerberg ([23]) in the context of dynamic doxastic logic, Board [5] in the context of multi-agent doxastic logic and van Benthem [3] in the context of dynamic epistemic logic (see also [2], [9] and the recent survey in [10]). More closely related to the analysis of this paper is Bonanno [6] where belief revision is studied within a temporal logic, which, on the semantic side, relies on branching-time frames where with each instant are associated two relations, one representing beliefs and the other representing information. As shown in [7], one can view such branching-time frames as a temporal generalization of the choice frames considered in this paper.

We conclude by noting that Corollary 9 can be viewed as analogous to the *frame characterization* results of modal logic (see, for example, [4]): given a finite rationalizable choice frame, every model based on it gives rise to a partial belief revision function which can be extended to a full AGM function and, conversely, a frame with this property must be rationalizable.

A Appendix

We shall prove Proposition 7 using Grove's [15] notion of system of spheres, which we recall below. As before, let A be an infinitely countable set of atoms and Φ the set of propositional formulas built on A. Let $\mathbb{M}$ be the set of maximally consistent sets of formulas for the propositional logic whose set of formulas is Φ. If $H \subseteq \Phi$ let $\mathbb{M}_H = \{x \in \mathbb{M} : H \subseteq x\}$. For a formula ϕ we shall write $\mathbb{M}_\phi$ instead of $\mathbb{M}_{\{\phi\}}$.

Definition 12 *A* system of spheres centered on $X \subseteq \mathbb{M}$ *is a collection $\mathcal{S}$ of subsets of $\mathbb{M}$ satisfying the following properties:*

(S.1) For all $U, V \in \mathcal{S}$, either $U \subseteq V$ or $V \subseteq U$;

(S.2) X is the smallest element of $\mathcal{S}$, that is, $X \in \mathcal{S}$ and, for every $U \in \mathcal{S}$, $X \subseteq U$;

(S.3) $\mathbb{M} \in \mathcal{S}$;

(S.4) if ϕ is a consistent formula, then there exists a smallest sphere in $\mathcal{S}$ denoted by $S(\phi)$ intersecting $\mathbb{M}_\phi$ (if ϕ is a contradiction, define $S(\phi)$ to be $\mathbb{M}$).

Notation 13 *For every $M \subseteq \mathbb{M}$, let $[M] = \{\phi \in \Phi : \phi \in x, \forall x \in M\}$.*

Theorem 14 *(Grove [15]) If $K \subseteq \Phi$ is a consistent belief set and $\mathcal{S}$ a system of spheres centered on $\mathbb{M}_K$ then the function $B_K^* : \Phi \to 2^\Phi$ defined by $B_K^*(\phi) = [S(\phi) \cap \mathbb{M}_\phi]$ is an AGM function based on K. Conversely, if B_K^* is an AGM function based on a consistent belief set K, then there exists a system of spheres $\mathcal{S}$ centered on $\mathbb{M}_K$ such that, for every formula ϕ, $B_K^*(\phi) = [S(\phi) \cap \mathbb{M}_\phi]$.*

Let $\langle \Omega, \mathcal{E}, f \rangle$ be a frame and V a valuation, giving rise to the model $\mathcal{M} = \langle \Omega, \mathcal{E}, f, V \rangle$. As noted in Footnote 6, we can associate with $\mathcal{M}$ a function $m : \Omega \to \mathbb{M}$ as follows:

[12] I am grateful to two anonymous referees for pointing out these simpler proofs.

$m(\omega) = \{\phi \in \Phi : \omega \models \phi\}$.[13] For $F \subseteq \Omega$ let $m(F) = \{m(\omega) : \omega \in F\}$. Clearly, for every formula ϕ, $m(\|\phi\|) \subseteq \mathbb{M}_\phi$. By definition of K (see (1)), $K = [m(f(\Omega))]$; furthermore, for every formula $\phi \in \Phi$, if $\|\phi\| = E \in \mathcal{E}$, then

$$[m(f(E))] = B_K(\phi). \tag{4}$$

In fact, for every formula ψ, $\psi \in [m(f(E))] \Leftrightarrow \psi \in x, \forall x \in m(f(E)) \Leftrightarrow \psi \in m(\omega), \forall\omega \in f(E) \Leftrightarrow \omega \models \psi, \forall\omega \in f(E) \Leftrightarrow f(E) \subseteq \|\psi\| \Leftrightarrow \psi \in B_K(\phi)$.

Proof of Proposition 7.[14] Let $\langle\Omega, \mathcal{E}, f\rangle$ be a finite rationalizable frame and let R be a total pre-order on Ω such that, for every $E \in \mathcal{E}$, $f(E) = best_R\ E$. For every $\omega \in \Omega$, let $D(\omega) = \{\omega' \in \Omega : \omega' R \omega\}$ and let $\Delta = \{D(\omega) : \omega \in \Omega\}$. Since Ω is finite, Δ can be written as a sequence $\langle D_0, ..., D_n\rangle$ satisfying: (i) $D_0 = f(\Omega)$, (ii) $D_j \subset D_{j+1}$, for all $j = 0, ..., n-1$, (iii) $D_n = \Omega$ and (iv) for every $E \in \mathcal{E}$, since $f(E) = best_R\ E \neq \varnothing$, $f(E) = E \cap D_{r(E)}$ where $r(E)$ is the smallest index $j \in \{0, ..., n\}$ such that $E \cap D_j \neq \varnothing$.[15] Let $\mathcal{S} = \{m(D_0), ..., m(D_n), \mathbb{M}\}$. We first show that $\mathcal{S}$ is a system of spheres centered on $m(D_0) = m(f(\Omega))$. Since, $D_j \subset D_{j+1}$ it is clear that $m(D_j) \subseteq m(D_{j+1})$, for every $j = 0, ..., n-1$.[16] Thus *(S.1)* and *(S.2)* of Definition 12 are satisfied; *(S.3)* is satisfied by construction and *(S.4)* is satisfied because of finiteness of $\mathcal{S}$. It follows from Theorem 14 that the function $B_K^* : \Phi \to 2^\Phi$ defined by $B_K^*(\phi) = [S(\phi) \cap \mathbb{M}_\phi]$ is an AGM belief revision function. We need to show that B_K^* is an extension of $B_K(\phi)$, that is, that if $\|\phi\| \in \mathcal{E}$ then $[S(\phi) \cap \mathbb{M}_\phi] = B_K(\phi)$. Let ϕ be such that $\|\phi\| = E \in \mathcal{E}$, so that $m(E) \subseteq \mathbb{M}_\phi$. Since $f(E) = E \cap D_{r(E)}$, $S(\phi) = m(D_{r(E)})$, so that $m(f(E)) = m(E \cap D_{r(E)}) \subseteq S(\phi) \cap \mathbb{M}_\phi$. Thus $[S(\phi) \cap \mathbb{M}_\phi] \subseteq [m(f(E))]$ and, therefore, by (4), $[S(\phi) \cap \mathbb{M}_\phi] \subseteq B_K(\phi)$. For the converse, let $\psi \in B_K(\phi)$; then $f(E) \subseteq \|\psi\|$. Suppose that $\psi \notin [S(\phi) \cap \mathbb{M}_\phi]$. Then there exists an $x \in S(\phi) \cap \mathbb{M}_\phi$ such that $\psi \notin x$. Since $S(\phi) = m(D_{r(E)})$, there exists an $\omega \in D_{r(E)}$ such that $x = m(\omega)$. Since $x \in \mathbb{M}_\phi$ and $\|\phi\| = E$, $\omega \in E$. Thus $\omega \in E \cap D_{r(E)} = f(E)$ so that, since $f(E) \subseteq \|\psi\|$, $\omega \models \psi$, that is, $\psi \in x = m(\omega)$, yielding a contradiction. ■

We now turn to the proof of Proposition 8. For the case where the frame $\langle\Omega, \mathcal{E}, f\rangle$ is such that $\mathcal{E}$ is a countable set, Proposition 8 can also be proved using Grove's approach. First of all, as Grove ([15], p. 160) notes, a system of spheres centered on X is equivalent to a total pre-order $\leqslant$ on $\mathbb{M}$ that satisfies the following properties:

(S$\leqslant$1) $X = \min_\leqslant \mathbb{M}$ (where, for every $T \subseteq \mathbb{M}$, $\min_\leqslant T = \{x \in T : x \leqslant y, \forall y \in T\}$),

(S$\leqslant$2) if ϕ is a consistent formula, then $\min_\leqslant \mathbb{M}_\phi \neq \varnothing$.

Theorem 14 can thus be restated in terms of a total pre-order $\leqslant$ on $\mathbb{M}$. We shall make use of the following related result in Gärdenfors and Rott ([14], Theorem 4.4.1, p. 79), which, in turn, is based on a result of Katsuno and Mendelzon ([18], Theorem 3.3, p. 269).[17]

Theorem 15 *(Gärdenfors and Rott [14]) $B_K^* : \Phi \to 2^\Phi$ is an AGM belief revision function based on a consistent belief set K if and only if there exists a total pre-order $\leqslant$ on $\mathbb{M}$ such that, (1) $\mathbb{M}_K = \min_\leqslant \mathbb{M}$ and (2) for every formula ϕ, $\mathbb{M}_{B_K^*(\phi)} = \min_\leqslant \mathbb{M}_\phi$.*

When $\mathcal{E}$ is a countable set, the above theorem can be used to prove Proposition 8 by constructing a model where, for every $E \in \mathcal{E}$, there is an atom $p_E \in A$ such that $\|p_E\| = E$.

Proof of Proposition 8 when $\mathcal{E}$ is countable. Let $\langle\Omega, \mathcal{E}, f\rangle$ be an AGM-consistent frame where Ω and $\mathcal{E}$ are countable sets. Construct a model where, for every $\omega \in \Omega$ there is

[13] Note that the function m in general is not injective, that is, it is possible that $\omega \neq \omega'$ and $m(\omega) = m(\omega')$. Furthermore, in general, $m(\Omega)$ is a proper subset of $\mathbb{M}$.

[14] An alternative proof, that does not rely on Grove's notion of system of spheres, is given in [8].

[15] For a generalization of this see Freund ([11], Theorem 5, p. 246).

[16] Note that it could be that $m(D_j) = m(D_{j+1})$ for some, or even all, j.

[17] This result is also a consequence of the representation of rational inference relations by means of total preorders (see [19]) and the identity between AGM revisions and rational relations (see [13]).

an atom $p_\omega \in A$ such that $V(p_\omega) = \{\omega\}$ and for every $E \in \mathcal{E}\backslash\{\Omega\}$ there is an atom $p_E \in A$ such that $V(p_E) = E$. Let B_K be the associated (partial) belief revision function and B_K^* an AGM extension of B_K (it exists since the frame is AGM-consistent). By Theorem 15 there exists a total pre-order $\leqslant$ of $\mathbb{M}$ such that

$$\begin{aligned}&\text{(a) } \mathbb{M}_K = \min\nolimits_{\leqslant} \mathbb{M} \text{ and}\\ &\text{(b) for every } \phi \in \Phi,\ \mathbb{M}_{B_K^*(\phi)} = \min\nolimits_{\leqslant} \mathbb{M}_\phi.\end{aligned} \tag{5}$$

Define $R \subseteq \Omega \times \Omega$ as follows: $\omega R \omega'$ if and only if $m(\omega) \leqslant m(\omega')$ (recall that $m : \Omega \to \mathbb{M}$ is defined by $m(\omega) = \{\phi \in \Phi : \omega \models \phi\}$). First we show that $f(\Omega) = best_R\ \Omega$. Fix arbitrary $\omega \in f(\Omega)$ and $\omega' \in \Omega$. By definition of K (see (1)), $m(f(\Omega)) \subseteq \mathbb{M}_K$. Thus, by (a) of (5), $m(\omega) \leqslant m(\omega')$; hence $\omega R \omega'$. Since ω' was chosen arbitrarily, it follows that $\omega \in best_R\ \Omega$. Hence, since $\omega \in f(\Omega)$ was chosen arbitrarily, $f(\Omega) \subseteq best_R\ \Omega$. Suppose that $best_R\ \Omega \nsubseteq f(\Omega)$. Then there exists an $\alpha \in best_R\ \Omega$ such that $\alpha \notin f(\Omega)$. Hence $f(\Omega) \subseteq \|\neg p_\alpha\| = \Omega\backslash\{\alpha\}$, that is, $\neg p_\alpha \in K$ so that $\neg p_\alpha \in x$ for every $x \in \mathbb{M}_K$. Thus, since $\alpha \models p_\alpha$ (that is, $p_\alpha \in m(\alpha)$), $m(\alpha) \notin \mathbb{M}_K$. By definition of choice frame, $f(\Omega) \neq \varnothing$. Fix a $\beta \in f(\Omega)$. Then $m(\beta) \in \mathbb{M}_K$ and thus, by (a) of (5), since $m(\alpha) \notin \mathbb{M}_K$, it is not the case that $m(\alpha) \leqslant m(\beta)$. Hence it is not the case that $\alpha R \beta$, contradicting the hypothesis that $\alpha \in best_R\ \Omega$.

Now let $E \in \mathcal{E}$ with $E \neq \Omega$. We want to show that $f(E) = best_R\ E$. Fix arbitrary $\omega \in f(E) \subseteq E = \|p_E\|$ and $\omega' \in E$. Since $B_K(p_E) = \{\phi \in \Phi : f(E) \subseteq \|\phi\|\}$, $m(f(E)) \subseteq \mathbb{M}_{B_K(p_E)}$. Thus, since $m(E) \subseteq \mathbb{M}_{p_E}$, it follows from (b) of (5) that $m(\omega) \leqslant m(\omega')$ and thus $\omega R \omega'$. Hence $f(E) \subseteq best_R\ E$. Suppose that $best_R\ E \nsubseteq f(E)$. Then there exists an $\alpha \in best_R\ E$ such that $\alpha \notin f(E)$. Hence $f(E) \subseteq \|\neg p_\alpha\|$, that is, $\neg p_\alpha \in B_K(p_E)$ so that $\neg p_\alpha \in x$ for every $x \in \mathbb{M}_{B_K(p_E)}$. Thus, since $\alpha \models p_\alpha$ (that is, $p_\alpha \in m(\alpha)$), $m(\alpha) \notin \mathbb{M}_{B_K(p_E)}$. By definition of choice frame, $f(E) \neq \varnothing$. Fix a $\beta \in f(E)$. Then $m(\beta) \in \mathbb{M}_{B_K(p_E)}$ and thus, by (b) of (5), since $m(\alpha) \notin \mathbb{M}_{B_K(p_E)}$, it is not the case that $m(\alpha) \leqslant m(\beta)$. Hence it is not the case that $\alpha R \beta$, contradicting the hypothesis that $\alpha \in best_R\ E$. ■

The assumption that $\mathcal{E}$ is a countable set is restrictive. For example, it rules out the case where $\Omega = \mathbb{N}$ (the set of natural numbers) and $\mathcal{E} = 2^{\mathbb{N}}$. An alternative proof, which does not require the assumption that $\mathcal{E}$ is countable, is given below. This proof may be of independent interest, since it is not based on the notion of system of spheres. Instead it relies on Hansson's result (Proposition 6). We begin with several lemmas. This first lemma is well known and the proof is omitted (a proof is given in [8]). The second lemma does not require any restrictions on the sets Ω and $\mathcal{E}$.

Lemma 16 *Let $H \subseteq \Phi$ and $\psi \in \Phi$. Then, for every formula χ, $\chi \in [H \cup \{\psi\}]^{PL}$ if and only if $(\psi \to \chi) \in [H]^{PL}$.*

Lemma 17 *If $\langle \Omega, \mathcal{E}, f \rangle$ is an AGM consistent choice frame then it satisfies Arrow's Axiom: if $E, F \in \mathcal{E}$ are such that $E \subseteq F$ and $E \cap f(F) \neq \varnothing$ then $f(E) = E \cap f(F)$.*

Proof. Fix arbitrary $E, F \in \mathcal{E}$ such that $E \subseteq F$ and $E \cap f(F) \neq \varnothing$ and suppose that $f(E) \neq E \cap f(F)$. Then either $E \cap f(F) \nsubseteq f(E)$ or $f(E) \nsubseteq E \cap f(F)$.

Case 1: $E \cap f(F) \nsubseteq f(E)$. Let $\omega \in E \cap f(F)$ be such that $\omega \notin f(E)$. Construct a model based on this frame where, for some atoms p, q and r, $\|p\| = E$, $\|q\| = F$ and $\|r\| = \{\omega\}$. Let B_K be the associated (partial) belief revision function and B_K^* be an AGM extension of B_K (it exists since, by hypothesis, the frame is AGM consistent). Since $\|p\| = E \in \mathcal{E}$ and $f(E) \subseteq E \subseteq F = \|q\|$, $q \in B_K(p)$ and thus $q \in B_K^*(p)$. Since p is not a contradiction, byAGM5 $B_K^*(p)$ is consistent. Thus $\neg q \notin B_K^*(p)$ and therefore, by AGM7 and AGM8,

$$B_K^*(p \wedge q) = [B_K^*(p) \cup \{q\}]^{PL} = [B_K^*(p)]^{PL} =_{\text{(by AGM1)}} B_K^*(p) \tag{6}$$

Since $\|q\| = F \in \mathcal{E}$, $\|p\| = E$ and $E \cap f(F) \neq \varnothing$, $\neg p \notin B_K(q) = B_K^*(q)$. Thus, by AGM7 and AGM8, $B_K^*(q \wedge p) = [B_K^*(q) \cup \{p\}]^{PL}$. It follows from this and (6) that

$$B_K^*(p) = [B_K^*(q) \cup \{p\}]^{PL}. \tag{7}$$

Since $\omega \notin f(E)$ and $\|r\| = \{\omega\}$, $f(E) \subseteq \|\neg r\|$, that is, $\neg r \in B_K(p)$ and hence (since B_K^* is an extension of B_K)

$$\neg r \in B_K^*(p). \tag{8}$$

Since $\omega \in E$, $\omega \in \|r \wedge p\|$. Thus, since $\omega \in f(F)$, $f(F) \cap \|r \wedge p\| \neq \varnothing$ so that $f(F) \nsubseteq \|\neg (r \wedge p)\| = \|(p \to \neg r)\|$ and thus, $(p \to \neg r) \notin B_K(q) = B_K^*(q)$. Hence, by Lemma 16, $\neg r \notin [B_K^*(q) \cup \{p\}]^{PL}$. This, together with (7), contradicts (8).

Case 2: $f(E) \nsubseteq E \cap f(F)$. Let $\omega \in f(E)$ be such that $\omega \notin E \cap f(F)$. Since $f(E) \subseteq E$, $\omega \in E$. Thus $\omega \notin f(F)$. As before, construct a model where, for some atoms p, q and r, $\|p\| = E$, $\|q\| = F$ and $\|r\| = \{\omega\}$. A repetition of the argument used above (leading to (6), making use of the hypotheses that $E \subseteq F$ and $E \cap f(F) \neq \varnothing$) yields (7). Since $\omega \notin f(F)$ and $\|r\| = \{\omega\}$, $f(F) \subseteq \|\neg r\|$ and thus, since $\|q\| = F$, $\neg r \in B_K(q) = B_K^*(q)$, which implies that

$$\neg r \in [B_K^*(q) \cup \{p\}]^{PL}. \tag{9}$$

On the other hand, since $\omega \in f(E)$ and $\|p\| = E$, $\neg r \notin B_K(p) = B_K^*(p)$. This, together with (7), contradicts (9). ■

Definition 18 *Let $\mathcal{F} = \langle \Omega, \mathcal{E}, f \rangle$ and $\mathcal{F}^+ = \langle \Omega, \mathcal{E}^+, f^+ \rangle$ be two choice frames. We say that $\mathcal{F}^+$ is an* extension *of $\mathcal{F}$ if $\mathcal{E} \subseteq \mathcal{E}^+$ and, for every $E \in \mathcal{E}$, $f^+(E) = f(E)$.*

Lemma 19 *Let $\mathcal{F} = \langle \Omega, \mathcal{E}, f \rangle$ be an AGM-consistent choice frame where Ω is a (possibly infinite) countable set. Let $\langle E_1, ..., E_n, E_{n+1} \rangle$ be a sequence in $\mathcal{E}$ such that $E_{n+1} = E_1$ and, $\forall k = 1, ..., n$, $E_k \cap f(E_{k+1}) \neq \varnothing$. Let $G = E_1 \cup ... \cup\ E_n$. Then there exists an extension $\mathcal{F}^+ = \langle \Omega, \mathcal{E}^+, f^+ \rangle$ of $\mathcal{F}$ such that (i) $\mathcal{E}^+ = \mathcal{E} \cup \{G\}$, and (ii) $\forall k = 1, ..., n$, if $E_k \cap f^+(G) \neq \varnothing$ then $f^+(E_k) = E_k \cap f^+(G)$. Furthermore, there exists a $j \in \{1, ..., n\}$ such that $E_j \cap f^+(G) \neq \varnothing$.*

Proof. Let $\mathcal{F} = \langle \Omega, \mathcal{E}, f \rangle$ be an AGM-consistent frame where Ω is countable and fix an arbitrary sequence $E_1, ..., E_{n+1}$ in $\mathcal{E}$ such that $E_{n+1} = E_1$ and, $\forall k = 1, ..., n$, $E_k \cap f(E_{k+1}) \neq \varnothing$. Let $G = E_1 \cup ... \cup\ E_n$. If $G \in \mathcal{E}$, then the result follows from Lemma 17 (take $\mathcal{F}^+ = \mathcal{F}$ and apply Arrow's Axiom). Suppose, therefore, that $G \notin \mathcal{E}$. Construct a model based on this frame where for every state $\omega \in \Omega$ there is an atom p_ω such that $\|p_\omega\| = \{\omega\}$. Furthermore, for every $i \in \{1, ..., n\}$, let p_i be an atom such that $\|p_i\| = E_i$. Let B_K be the associated (partial) belief revision function and let B_K^* be an AGM extension of B_K (it exists since, by hypothesis, $\mathcal{F}$ is AGM-consistent). Let

$$J = \{i \in \{1, ..., n\} : \neg p_i \notin B_K^*(p_1 \vee ... \vee p_n)\}. \tag{10}$$

First we show that $J \neq \varnothing$. Suppose that $J = \varnothing$. Then, for every $i \in \{1, ..., n\}$, $\neg p_i \in B_K^*(p_1 \vee ... \vee p_n)$. Hence, since (by AGM1) $B_K^*(p_1 \vee ... \vee p_n)$ is deductively closed, $(\neg p_1 \wedge ... \wedge \neg p_n) \in B_K^*(p_1 \vee ... \vee p_n)$. Thus, since $(\neg p_1 \wedge ... \wedge \neg p_n)$ is equivalent to $\neg(p_1 \vee ... \vee p_n)$,

$$\neg(p_1 \vee ... \vee p_n) \in B_K^*(p_1 \vee ... \vee p_n). \tag{11}$$

By AGM2,

$$(p_1 \vee ... \vee p_n) \in B_K^*(p_1 \vee ... \vee p_n) \tag{12}$$

Since $(p_1 \vee ... \vee p_n)$ is not a contradiction, by AGM5, $B_K^*(p_1 \vee ... \vee p_n)$ is consistent, so that, for every formula ϕ, it is not the case that both ϕ and $\neg\phi$ belong to $B_K^*(p_1 \vee ... \vee p_n)$, contradicting (11) and (12). Thus $J \neq \varnothing$.

Define $\langle \Omega, \mathcal{E}^+, f^+ \rangle$ as follows: $\mathcal{E}^+ = \mathcal{E} \cup \{G\}$ (where $G = E_1 \cup ... \cup E_n$); for every $E \in \mathcal{E}$, $f^+(E) = f(E)$ and

$$f^+(G) = \bigcup_{j \in J} f(E_j). \tag{13}$$

Fix an arbitrary $k \in \{1, ..., n\}$ and suppose that $E_k \cap f^+(G) \neq \varnothing$. We want to show that $f^+(E_k) = E_k \cap f^+(G)$ (note that, by definition of f^+, $f^+(E_k) = f(E_k)$).

First we show that $k \in J$. Suppose that $k \notin J$, that is,

$$\neg p_k \in B_K^*(p_1 \vee ... \vee p_n). \tag{14}$$

Since, by hypothesis, $E_k \cap f^+(G) \neq \varnothing$, $f^+(G) \neq \varnothing$ and thus, by (13), there exists an $s \in J$ such that $E_k \cap f(E_s) \neq \varnothing$. Thus $f(E_s) \nsubseteq \Omega \backslash E_k = \|\neg p_k\|$ and therefore $\neg p_k \notin B_K(p_s)$. Since $\|p_s\| = E_s \in \mathcal{E}$, $B_K(p_s) = B_K^*(p_s)$. Thus

$$\neg p_k \notin B_K^*(p_s). \tag{15}$$

Since $s \in J$, $\neg p_s \notin B_K^*(p_1 \vee ... \vee p_n)$. Thus, by AGM7 and AGM8 (and noting that p_s is equivalent to $((p_1 \vee ... \vee p_n) \wedge p_s)$),

$$B_K^*(p_s) = [B_K^*(p_1 \vee ... \vee p_n) \cup \{p_s\}]^{PL}. \tag{16}$$

It follows from (15), (16) and Lemma 16 (and the fact that, by AGM1, $B_K^*(p_s) = [B_K^*(p_s)]^{PL}$) that

$$(p_s \rightarrow \neg p_k) \notin B_K^*(p_1 \vee ... \vee p_n). \tag{17}$$

Since $(\neg p_k \rightarrow (p_s \rightarrow \neg p_k))$ is a tautology and (by AGM1) $B_K^*(p_1 \vee ... \vee p_n)$ is deductively closed, $(\neg p_k \rightarrow (p_s \rightarrow \neg p_k)) \in B_K^*(p_1 \vee ... \vee p_n)$. It follows from this and (14) that $(p_s \rightarrow \neg p_k) \in B_K^*(p_1 \vee ... \vee p_n)$, contradicting (17).

Thus we have shown that $k \in J$, that is,

$$\neg p_k \notin B_K^*(p_1 \vee ... \vee p_n). \tag{18}$$

Hence, by (13), $f(E_k) \subseteq f^+(G)$. Furthermore, by definition of choice frame, $f(E_k) \subseteq E_k$. Thus $f(E_k) \subseteq E_k \cap f^+(G)$.

Now we show the converse, namely that $E_k \cap f^+(G) \subseteq f(E_k)$. Suppose not. Then there exists an $\alpha \in E_k \cap f^+(G)$ such that $\alpha \notin f(E_k)$, that is, $f(E_k) \subseteq \|\neg p_\alpha\|$, so that

$$\neg p_\alpha \in B_K(p_k) = B_K^*(p_k). \tag{19}$$

By (18) and AGM7 and AGM8 (and noting that $(p_1 \vee ... \vee p_n) \wedge p_k$ is equivalent to p_k), $B_K^*(p_k) = [B_K^*(p_1 \vee ... \vee p_n) \cup \{p_k\}]^{PL}$. Hence, by (19) and Lemma 16,

$$(p_k \rightarrow \neg p_\alpha) \in B_K^*(p_1 \vee ... \vee p_n). \tag{20}$$

Since $\alpha \in f^+(G)$, there exists an $s \in J$ such that $\alpha \in f(E_s)$. Thus $\neg p_\alpha \notin B_K(p_s) = B_K^*(p_s)$. Furthermore, since $s \in J$, $\neg p_s \notin B_K^*(p_1 \vee ... \vee p_n)$ (see (10)), so that (16) holds and, therefore, by Lemma 16,

$$(p_s \to \neg p_\alpha) \notin B_K^*(p_1 \vee ... \vee p_n). \tag{21}$$

Since $\alpha \in E_k$, $\{\alpha\} = \|p_\alpha\| \subseteq \|p_k\| = E_k$, that is, $\|p_\alpha \to p_k\| = \Omega$. Hence $f(E_s) \subseteq \|p_\alpha \to p_k\|$, that is, $(p_\alpha \to p_k) \in B_K(p_s) = B_K^*(p_s)$. Thus, by (16) and Lemma 16,

$$(p_s \to (p_\alpha \to p_k)) \in B_K^*(p_1 \vee ... \vee p_n). \tag{22}$$

Since $(p_s \to (p_\alpha \to p_k))$ is equivalent to $(p_s \wedge p_\alpha \to p_k)$, it follows from (20), (22) and the fact that (by AGM1) $B_K^*(p_1 \vee ... \vee p_n)$ is deductively closed, that $((p_s \wedge p_\alpha \to p_k) \wedge (p_k \to \neg p_\alpha)) \in B_K^*(p_1 \vee ... \vee p_n)$ so that $(p_s \wedge p_\alpha \to \neg p_\alpha) \in B_K^*(p_1 \vee ... \vee p_n)$. But $(p_s \wedge p_\alpha \to \neg p_\alpha)$ is equivalent to $(p_s \to \neg p_\alpha)$. Hence $(p_s \to \neg p_\alpha) \in B_K^*(p_1 \vee ... \vee p_n)$, contradicting (21). Thus we have shown that $f(E_k) = E_k \cap f^+(G)$; since $f(E_k) = f^+(E_k)$, it follows that $f^+(E_k) = E_k \cap f^+(G)$.

It only remains to show that there exists a $j \in \{1, ..., n\}$ such that $E_j \cap f^+(G) \neq \varnothing$. Since $f^+(G) = \bigcup_{j \in J} f(E_j)$, it follows that there exists a $j \in \{1, ..., n\}$ such that $f(E_j) = f(E_j) \cap f^+(G)$. By definition of choice frame, $\varnothing \neq f(E_j) \subseteq E_j$. Thus $E_j \cap f^+(G) \neq \varnothing$. ■

Lemma 20 *Let $\mathcal{F} = \langle \Omega, \mathcal{E}, f \rangle$ be a choice frame and let $\langle E_1, ..., E_n, E_{n+1} \rangle$ be a sequence in $\mathcal{E}$ such that $E_{n+1} = E_1$ and, $\forall k = 1, ..., n$, $E_k \cap f(E_{k+1}) \neq \varnothing$. Let $G = E_1 \cup ... \cup E_n$ and let $\mathcal{F}^+ = \langle \Omega, \mathcal{E}^+, f^+ \rangle$ be an extension of $\mathcal{F}$ such that (i) $\mathcal{E}^+ = \mathcal{E} \cup \{G\}$ and (ii) $\forall k = 1, ..., n$, if $E_k \cap f^+(G) \neq \varnothing$ then $f^+(E_k) = E_k \cap f^+(G)$. Then, for every $k = 1, ..., n$*

$$\text{if } E_{k+1} \cap f^+(G) \neq \varnothing \text{ then } \begin{cases} E_k \cap f^+(G) \neq \varnothing \\ \text{and} \\ E_k \cap f(E_{k+1}) = f(E_k) \cap E_{k+1}. \end{cases} \tag{23}$$

Proof. Fix a $k \in \{1, ..., n\}$ and assume that $E_{k+1} \cap f^+(G) \neq \varnothing$. Then, by hypothesis (ii), $f(E_{k+1}) = E_{k+1} \cap f^+(G)$. Thus

$$E_k \cap f(E_{k+1}) = E_k \cap E_{k+1} \cap f^+(G). \tag{24}$$

By hypothesis, $E_k \cap f(E_{k+1}) \neq \varnothing$. Thus, by (24), $E_k \cap E_{k+1} \cap f^+(G) \neq \varnothing$ so that

$$E_k \cap f^+(G) \neq \varnothing. \tag{25}$$

It follows from (25) and hypothesis (ii) that $f(E_k) = E_k \cap f^+(G)$, so that

$$f(E_k) \cap E_{k+1} = E_k \cap E_{k+1} \cap f^+(G). \tag{26}$$

From (24) and (26) we get that $E_k \cap f(E_{k+1}) = f(E_k) \cap E_{k+1}$. ■

Corollary 21 *Let $\mathcal{F} = \langle \Omega, \mathcal{E}, f \rangle$ be a choice frame and let $\langle E_1, ..., E_n, E_{n+1} \rangle$ be a sequence in $\mathcal{E}$ such that $E_{n+1} = E_1$ and, $\forall k = 1, ..., n$, $E_k \cap f(E_{k+1}) \neq \varnothing$. Let $G = E_1 \cup ... \cup E_n$ and let $\mathcal{F}^+ = \langle \Omega, \mathcal{E}^+, f^+ \rangle$ be an extension of $\mathcal{F}$ such that (i) $\mathcal{E}^+ = \mathcal{E} \cup \{G\}$, (ii) $\forall k = 1, ..., n$, if $E_k \cap f^+(G) \neq \varnothing$ then $f^+(E_k) = E_k \cap f^+(G)$ and (iii) there exists a $j \in \{0, ..., n-1\}$ such that $E_{j+1} \cap f^+(G) \neq \varnothing$. Then, for every $k = 1, ..., n$, $E_k \cap f(E_{k+1}) = f(E_k) \cap E_{k+1}$.*

Proof. By Lemma 20, $E_j \cap f(E_{j+1}) = f(E_j) \cap E_{j+1}$ and $E_j \cap f^+(G) \neq \varnothing$. Thus applying the lemma again we get $E_{j-1} \cap f(E_j) = f(E_{j-1}) \cap E_j$ (taking $j-1 = n$ if $j = 1$). Repeating this argument $n-1$ times (interpreting $j-r$ as $n-(j-r)$ if $j-r < 1$) yields the desired result. ■

Proof of Proposition 8. Let $\mathcal{F} = \langle \Omega, \mathcal{E}, f \rangle$ be an AGM-consistent choice frame where Ω is a (possibly infinite) countable set. Let $\langle E_1, ..., E_n, E_{n+1} \rangle$ be a sequence in $\mathcal{E}$ such that

$E_{n+1} = E_1$ and, $\forall k = 1, ..., n$, $E_k \cap f(E_{k+1}) \neq \varnothing$. By Lemma 19 and Corollary 21, for every $k = 1, ..., n$, $E_k \cap f(E_{k+1}) = f(E_k) \cap E_{k+1}$. It follows from Proposition 6 that $\langle \Omega, \mathcal{E}, f \rangle$ is rationalizable. ■

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