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OLIGOPOLY EQUILIBRIA WHEN FIRMS HAVE LOCAL KNOWLEDGE OF DEMAND*

BY GIACOMO BONANNO[1]

1. INTRODUCTION

The notion of Nash equilibrium in static oligopoly games is based on the assumption that each firm knows its entire demand curve (and, therefore, its entire profit function). It is much more likely, however, that firms only have some idea of the outcome of small price variations within some relatively small interval of prices. This is because firms can only learn their demand functions through price experiments and if they are risk-averse and/or have a low discount factor, they will be unwilling to engage in extensive price experiments involving large variations in price.[2] We can therefore expect firms to experiment through small price variations and stop when they reach a price such that no small deviation from it brings about an increase in profits.[3] In other words, firms will stop experimenting when they reach a local maximum of their profit functions.[4] The purpose of this paper is to investigate the existence of local Nash equilibria. In other words, this paper investigates a vector of prices such that each firm is at a local maximum of its profit function, given the prices charged by the other firms (obviously, a Nash equilibrium is a local Nash equilibrium, but the converse is not true, as the first example in Section 2 shows).

It is well known (Roberts and Sonnenschein 1977) that Nash equilibria may fail to exist unless one imposes restrictions on the demand functions (e.g. concavity) which cannot be justified on the basis of standard assumptions on consumers' preferences.[5] Bonanno and Zeeman (1985) recently showed that if firms

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[1] This paper draws from Chapter 2 of my Ph.D. thesis which was submitted to the London School of Economics in May 1985. I am grateful to Oliver Hart for his help and encouragement and to Christopher Zeeman and an anonymous referee for constructive comments and suggestions.

[2] Indeed, it was recently shown by Aghion, Bolton, and Jullien (1986) that if the discount factor is sufficiently small even a risk-neutral firm may be very reluctant to engage in (even very small) price experiments.

[3] This idea was first suggested by Baumol and Quandt (1964).

[4] Problems may arise when many interdependent firms experiment at the same time, in which case the demand curve of each single firm would be shifting during the price experiments. One way of overcoming this difficulty would be to restrict oneself to the case of monopolistic competition (large number of small firms), characterized by the fact that the action of any single firm has a negligible effect on the demand faced by every other firm (see, for example, Hart 1985). In this case the effects of the other firms' simultaneous experiments would be negligible and would, on average, cancel out.

[5] The problem posed by Roberts and Sonnenschein (1977) was named "an impossibility theorem" by Friedman (1982, p. 532).

perform local price experiments and extrapolate a linear demand function which is "correct" (in the sense that it coincides with the linear approximation to the "true" demand curve at that point) then an equilibrium exists always, that is, with arbitrary demand functions.[6] In this paper, we consider the case where firms perform enough local price experiments to enable them to arrive at a correct, rather than approximate, estimation of their demand curves; that is, they learn the correct shape of their demand functions locally.

The paper is organized as follows. In Section 2, we show, by means of an example, that there are oligopoly games which have no Nash equilibria but possess (isolated) local Nash equilibria and then give an example of a duopoly game which has no local Nash equilibria. In Section 3, we give sufficient conditions for the existence of a local Nash equilibrium.

2. EXAMPLES

Both examples in this section are of a duopoly with differentiated products and zero costs of production, where firms compete in prices. The first example shows that there are games which have no Nash equilibria but possess a finite number of local Nash equilibria. Let p_i be the price of firm i ($i = 1, 2$), $p = (p_1, p_2)$ and let $D_i(p)$ be the demand function of firm i. Let[7]

$$D_1(p) = -p_1^3 + 12p_1^2 - 52p_1 + 93 + p_2 \tag{1}$$

The profit function of firm 1 is given by

$$P_1(p) = p_1 D_1(p) \tag{2}$$

Figure 1 shows the set of points p such that $(\partial P_1/\partial p_1)(p) = 0$. The heavy lines represent global maxima of P_1 (as a function of p_1, parametrized by p_2), the continuous lines local maxima and the dashed line local minima (therefore the reaction curve of firm 1 is given by the union of the two heavy lines).

Let:[8]

$$\begin{aligned} D_2(p) &= 3 + 0.74p_1 - p_2 && \text{for } 0 \leq p_2 \leq 1 + 0.68p_1 \\ &= 2.5 + 0.4p_1 - 0.5p_2 && \text{for } p_2 \geq 1 + 0.68p_1 \end{aligned} \tag{3}$$

The profit function of firm 2 is given by $P_2(p) = p_2 D_2(p)$. Figure 2 shows the set of points p such that $(\partial P_2/\partial p_2)(p) = 0$. As in Figure 1, the heavy lines repre-

[6] This notion of equilibrium was first introduced by Silvestre (1977). Unlike Silvestre, however, Bonanno and Zeeman (1985) impose no restrictions at all on the demand curves (not even that they be decreasing everywhere). The use of the word "equilibrium" is justified, since—even though a firm may actually be at a local minimum of its "true" profit function—it will believe, on the basis of the estimated demand curve, that it is maximizing profits.

[7] To be more precise, we should say that D_1 is given by the maximum between zero and (1). The same applies to (3), (4), and (5). Note that $D_1 > 0$ implies $\partial D_1/\partial p_1 < 0$, $\partial D_1/\partial p_2 > 0$, and $|\partial D_1/\partial p_1| > \partial D_1/\partial p_2$.

[8] Note, again, that $D_2 > 0$ implies $\partial D_2/\partial p_1 > 0$, $\partial D_2/\partial p_2 < 0$, and $|\partial D_2/\partial p_2| > \partial D_2/\partial p_1$.

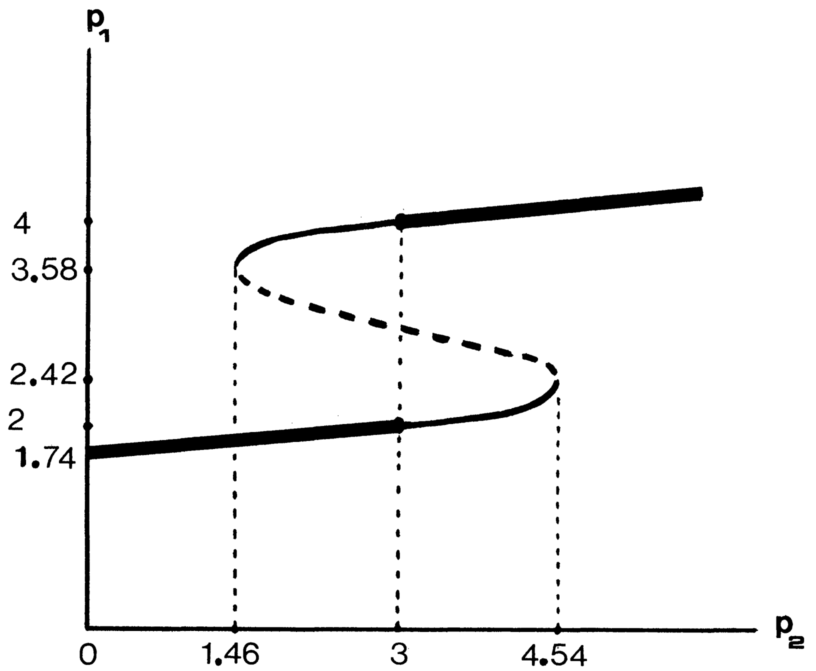

FIGURE 1

sent global maxima of P_2 (as a function of p_2, parametrized by p_1), the continuous lines local maxima and the dashed line local minima (therefore the reaction curve of firm 2 is given by the union of the two heavy lines).[9]

Figure 3 shows Figures 1 and 2 together. It can be seen that this game has no Nash equilibria, since the reaction curves do not intersect (recall that the reaction curve of firm 1 is given by the union of the heavy lines of the S-shaped curve, while the reaction curve of firm 2 is given by the union of the heavy lines of the (inverted) Z-shaped curve). However, there are four local Nash equilibria, denoted by *A*, *B*, *C*, and *D*.[10] At equilibria *A* and *B*, firm 2 is at a global maximum of its profit function, while firm 1 is at a local—but not global—maximum of its profit function. Conversely, at equilibria *C* and *D*, firm 1 is at a global maximum, while firm 2 is at a local—but not global—maximum of its profit function.

[9] While the global and local maxima of P_2 are smooth, the local minima are not, since they occur at the point $p_2 = 1 + 0.68p_1$ at which the demand function D_2 (given by (3)) has a kink. This fact is of no consequence for our analysis, since we are not interested in the local minima of the profit function P_2. The function (3) was chosen in order to simplify the figures, but it could easily be replaced by a smooth approximation (which would smooth the local minima of P_2). In fact, every continuous function can be approximated arbitrarily closely by a smooth function (see Hirsh 1976, Theorem 2.4, p. 47). A smooth approximation of (3) would also transform the Z-shaped curve of Figure 2 into a smooth (inverted) S-shaped curve.

[10] The coordinates of A, B, C, and D are respectively (2.0423, 3.3169), (3.9974, 2.9790), (4.1208, 4.1483), and (1.9127, 2.2077), where the first coordinate is p_1 and the second p_2.

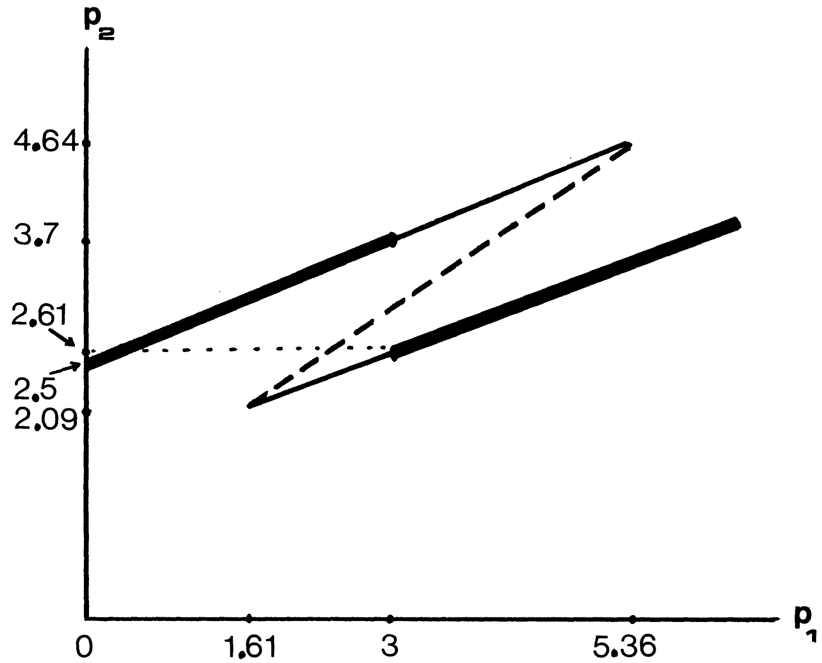

FIGURE 2

The next example shows that local Nash equilibria do not always exist. Let:

(4) $$D_1(p) = -0.0014p_1^3 + 0.0748p_1^2 - 1.4796p_1 + 0.5829 + p_2$$

and

(5) $$\begin{aligned} D_2(p) &= 10.5 + 1.9p_1 - 2p_2 \quad &\text{for} \quad 0 \le p_2 \le 0.33 + 0.94p_1 \\ &= 10 + 0.49p_1 - 0.5p_2 \quad &\text{for} \quad p_2 \ge 0.33 + 0.94p_1 \end{aligned}$$

Figure 4 shows the two curves defined by $(\partial P_1/\partial p_1)(p) = 0$ and $(\partial P_2/\partial p_2)(p) = 0$. As before, the heavy lines represent global maxima, the continuous lines represent local maxima, and the dashed lines represent local minima (of the corresponding profit functions). It can be seen from Figure 4 that this game has no local Nash equilibria.[11]

This last example shows that it is necessary to impose some restrictions on the demand curves in order to guarantee the existence of a local Nash equilibrium.

[11] At point E, both firms are at a local minimum of their profit functions. As explained in footnote 6, each firm will believe that it is maximizing profits if it only knows the linear approximation of its demand curve at that point (Bonanno-Zeeman 1985, p. 282).

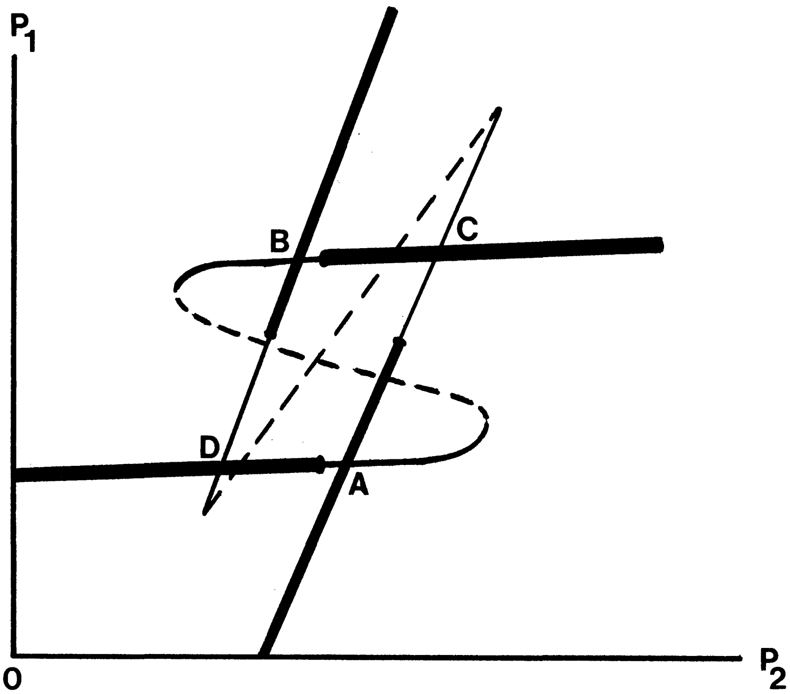

FIGURE 3

3. EXISTENCE OF LOCAL NASH EQUILIBRIA

We shall use the same model which was used by Bonanno and Zeeman (1985). There are n firms, indexed by $i = 1, \ldots, n$, producing differentiated products and competing in prices. The cost function of firm i is given by $C_i(q) = c_i q$, where c_i is a constant[12] and q denotes output. Let $p = (p_1, \ldots, p_n)$, where p_i is the price of firm i, and let $D_i(p)$ be the demand function of firm i. D_i is assumed to be continuous, to become zero at some price $p_i' > c_i$, which varies smoothly with the prices charged by the other firms and is bounded away from infinity (in Figure 5a, p_i^* represents an upper bound for p_i'). Finally, the partial derivative $\partial D_i/\partial p_i$ is assumed to be C^1 and to be negative at the point where the demand curve crosses the p_i-axis (the left derivative, to be precise).[13] Figure 5a shows a possible shape of D_i for a given vector of prices charged by the other firms (note that, contrary to common use, we measure price on the horizontal axis). Figure 5b shows the

[12] Fixed costs of production can also be allowed for.

[13] For a more detailed description of the model and discussion of the assumptions the reader is referred to Bonanno-Zeeman (1985). Here we have strengthened assumption 1(d) (p. 280) by requiring p_i' to be a C^2 function rather than merely continuous and assumption 3 (p. 281) by requiring D_i and $\partial D_i/\partial p_i$ to be C^1 rather than merely continuous. These assumptions are still weaker than those which are normally made in oligopoly models with differentiated products (Friedman 1982, p. 501).

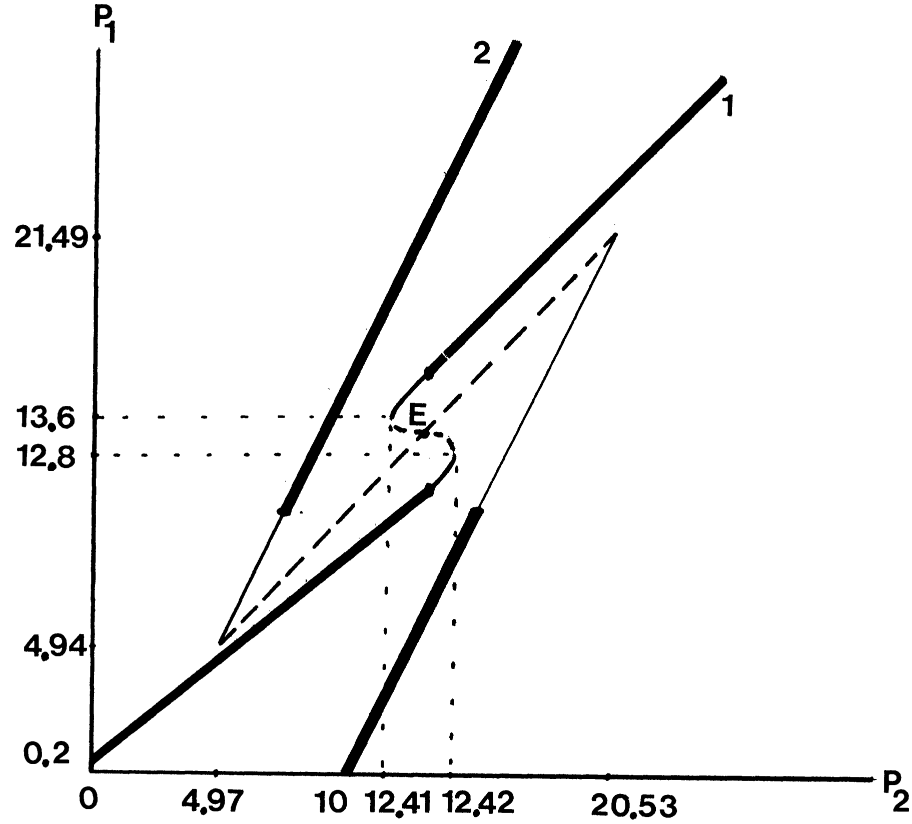

FIGURE 4

corresponding profit function

(6) $$P_i(p) = (p_i - c_i)D_i(p)$$

(the function $\bar{P}_i$ will be defined later).

We can now introduce the following definition.

DEFINITION 1. *A local Nash equilibrium is a vector of prices p° such that*

(7) $$P_i(p^\circ) > 0 \qquad i = 1, \ldots, n$$

and for each i there exists a neighborhood N_i of p_i° such that

(8) $$P_i(p^\circ) \geq P_i(p_i, p_{-i}^\circ) \qquad \textit{for all } p_i \in N_i$$

(where (p_i, p_{-i}°) denotes the vector p° with p_i° replaced by p_i).

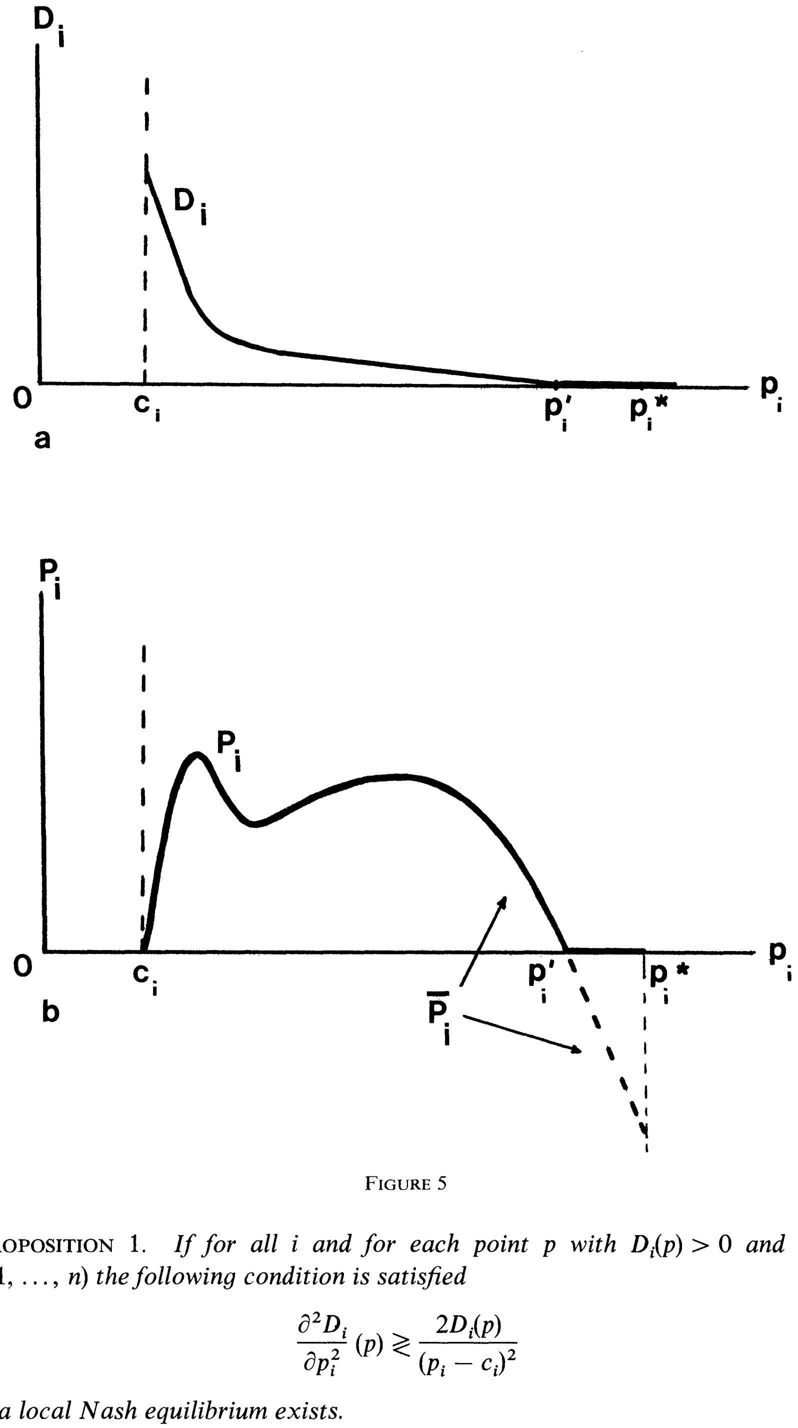

FIGURE 5

PROPOSITION 1. *If for all i and for each point p with* $D_i(p) > 0$ *and* $p_i > c_i$ $(i = 1, \ldots, n)$ *the following condition is satisfied*

$$\frac{\partial^2 D_i}{\partial p_i^2}(p) \gtrless \frac{2D_i(p)}{(p_i - c_i)^2} \tag{9}$$

then a local Nash equilibrium exists.

Note that, since the RHS of (9) is positive, the above proposition implies that when the demand curve is concave ($\partial^2 D_i/\partial p_i^2 \leq 0$) or the convexities of D_i are small, then a local Nash equilibrium exists. Note, however, that the proposition also implies that a local Nash equilibrium exists even if the convexities of the demand function are very large (the RHS of (9) tends to $+\infty$ as p_i approaches c_i from above).

A proof of the above proposition is given in the Appendix. Condition (9) implies that there is no point p such that $(P_i(p) > 0)$ and $(\partial P_i/\partial p_i)(p) = (\partial^2 P_i/\partial p_i^2)(p) = 0$; that is, it is not the case that—as the prices of the other firms vary—a local maximum of the profit function of firm i gets closer and closer to a local minimum and eventually coalesces with it, giving rise to a point of inflection.

Note that, just as quasiconcavity of the profit functions is a sufficient but not necessary condition for the existence of Nash equilibria, so condition (9) is sufficient but not necessary for the existence of local Nash equilibria. In both the examples of Section 2, condition (9) is not satisfied; however, only in the second example does the failure of condition (9) to hold give rise to non-existence of local Nash equilibria.[14]

4. Conclusion

The first example of Section 2 shows that the class of oligopoly games which have local Nash equilibria is larger than that of games which have Nash equilibria.[15] Proposition 1 gives sufficient conditions for the existence of local Nash equilibria which are much weaker than those needed to prove the existence of Nash equilibria (quasi-concavity of the profit functions). However, the second example of Section 2 shows that, unlike the equilibria studied by Bonanno and Zeeman (1985), local Nash equilibria may not exist if one allows for arbitrary demand functions. Since Roberts and Sonnenschein (1977) effectively showed that standard conditions on consumers' preferences (e.g. convexity) are compatible with demand functions of any shape, the second example of Section 2 could also be interpreted as "an impossibility theorem" (cf. Friedman 1982, p. 532) for local Nash equilibria, while the proposition is the analogue (for local Nash equilibria) of the standard existence theorem for Nash equilibria (it is also worth noting that

[14] In the first example there are two points (p_1, p_2) at which $\partial^2 D_1/\partial p_1^2 = 2D_1/p_1^2$ (recall that in both examples costs of production are zero) and these are (2.42, 4.54) and (3.58, 1.46) (see Figure 1). Similarly, any smooth approximation of D_2 (see footnote 9) would violate condition (9) at two points (p_1, p_2) close to (1.61, 2.09) and (5.36, 4.64) (see Figure 2). In the second example, D_1 fails to satisfy condition (9) at the two points (12.8, 12.42) and (13.6, 12.41) and any smooth approximation of D_2 would fail to satisfy condition (9) at two points close to (4.94, 4.97) and (21.49, 20.53). The above eight points are points of inflection of the corresponding profit functions.

[15] It can also be shown that, generically, local Nash equilibria are isolated and finite in number and that the property of having a local Nash equilibrium is structurally stable (see Bonanno 1985, Proposition 2.4, p. 69).

if the profit functions are quasiconcave, every local Nash equilibrium is a Nash equilibrium).

University of California at Davis, U.S.A.

APPENDIX

PROOF OF PROPOSITION 1. Proposition 1 is a consequence of the following lemma. Let C be the n-dimensional cube

$$C = \{x \in R^n / -1 \leq x_i \leq 1,\ i = 1, \ldots, n\}$$

and let f_i: $C \to R$ $(i = 1, \ldots, n)$ be n continuous functions satisfying the following hypotheses:

(A.1) $$\partial f_i / \partial x_i \quad \text{exists and is } C^1$$

(A.2) $$\partial f_i / \partial x_i > 0 \quad \text{if} \quad x_i = -1$$
$$< 0 \quad \text{if} \quad x_i = +1.$$

LEMMA. *If for each i and x*

(A.3) $$(\partial f_i / \partial x_i)(x) = 0 \quad \text{implies} \quad (\partial^2 f_i / \partial x_i^2)(x) \neq 0$$

then there exists a point x° in the interior of C such that for each i

$$(\partial f_i / \partial x_i)(x^\circ) = 0 \quad \text{and} \quad (\partial^2 f_i / \partial x_i^2)(x^\circ) < 0$$

that is, x° is a local Nash equilibrium of the game with payoff functions f_i $(i = 1, \ldots, n)$.

PROOF OF LEMMA. For each i let

$$S_i = \{x \in C / (\partial f_i / \partial x_i)(x) = 0\}.$$

Since $\partial f_i / \partial x_i$ is continuous and C is compact, S_i is a compact set and therefore has a finite number of connected components. Let K be the $(n - 1)$-dimensional cube

$$K = \{y \in R^{n-1} / -1 \leq y_j \leq 1,\ j = 1, \ldots, n - 1\}.$$

We shall denote a point $x \in C$ by $x = (y, x_i)$ with $y \in K$ and $x_i \in [-1, 1]$. We first want to show that each connected component of S_i projects onto K (that is, if $M \subset S_i$ is a connected component of S_i, then for each $y \in K$ there exists an $x_i \in [-1, 1]$ such that $(y, x_i) \in M$). First of all, the boundary conditions (A.2) ensure that S_i itself projects onto K (that is, for each $y \in K$ there exists an $x_i \in [-1, 1]$ s.t. $(\partial f_i / \partial x_i)(y, x_i) = 0$). Furthermore, by (A.3), for each $y \in K$ the set

$$\{x_i \in [-1, 1] / (\partial f_i / \partial x_i)(y, x_i) = 0\}$$

is a finite set (since it is a discrete subset of a compact set). Therefore, by the implicit function theorem (applied to the function $\partial f_i/\partial x_i$: $C = K \times [-1, 1] \to R$) for each $y \in K$ there exists an open neighborhood N_y of y in K such that S_i is locally diffeomorphic to the Cartesian product of N_y and a finite set. More precisely, fix an arbitrary $\bar{y} \in K$ and let $x_i^{(r)} \in [-1, 1]$, $r = 1, \ldots, m$, be all the solutions to the equation $(\partial f_i/\partial x_i)(\bar{y}, x_i) = 0$ (in general, the number of solutions, m, could vary with $y \in K$). By the implicit function theorem, for every solution $x_i^{(r)}$ there exists a neighborhood $N_{\bar{y}}^{(r)}$ of $\bar{y}$ in K, a neighborhood $W^{(r)}$ of $(\bar{y}, x_i^{(r)})$ in $C = K \times [-1, 1]$ and a C^1 function $h^{(r)}$: $N_{\bar{y}}^{(r)} \to R$ such that:

(i) $(\partial f_i/\partial x_i)(y, h^{(r)}(y)) = 0$ for all $y \in N_{\bar{y}}^{(r)}$

(ii) The only solutions to the equation $(\partial f_i/\partial x_i)(y, x_i) = 0$ in $W^{(r)}$ are those given by (i) (see, for example, Field 1976, p. 203).

Then there exists an open neighborhood $N_{\bar{y}}$ of $\bar{y}$ in K (contained in the intersection of the neighborhoods $N_{\bar{y}}^{(r)}$, $r = 1, \ldots, m$) with the following property: if for some $x_i \in [-1, 1]$ and for some $y \in N_{\bar{y}}$, $(y, x_i) \in S_i$ then $x_i = h^{(r)}(y)$ for some $r = 1, \ldots, m$. If not, there would exist a sequence of points $\langle y^k \rangle$ in K converging to $\bar{y}$ and a sequence of points $\langle x_i^k \rangle$ in $[-1, 1]$ such that for all k, $(y^k, x_i^k) \in S_i$ and $(y^k, x_i^k) \notin W^{(r)}$ for all $r = 1, \ldots, m$. Since $[-1, 1]$ is compact, the sequence $\langle x_i^k \rangle$ has a convergent subsequence. Let $\bar{x}_i$ be the limit of this subsequence. Then since S_i is closed, $(\bar{y}, \bar{x}_i) \in S_i$. Then $\bar{x}_i = x_i^{(r)}$ for some r ($r = 1, \ldots, m$), which implies that $(y^k, x_i^k) \in W^{(r)}$ for k sufficiently large (where the superscript k now refers to the convergent subsequence), yielding a contradiction.

The set $\{N_y\}_{y \in K}$ is an open covering of K, which is a compact set. Therefore, there exists a finite subcovering $Z = \{N_{y_1}, N_{y_2}, \ldots, N_{y_p}\}$. N_{y_1} has a nonempty intersection with another N_{y_j}, $j \neq 1$ (otherwise the points in K which belong to the boundary of N_{y_1} would not belong to any of the open sets in Z, contradicting the fact that Z is a covering of K). Thus the number of solutions to the equation $(\partial f_i/\partial x_i)(y, x_i) = 0$ with fixed y is the same for all $y \in N_{y_1} \cup N_{y_j}$ and every point $(y, x_i) \in S_i$, with $y \in N_{y_1} \cup N_{y_j}$ (and $x_i \in [-1, 1]$), is situated on the graph of a C^1 function h: $N_{y_1} \cup N_{y_j} \to [-1, 1]$).

Similarly, $N_{y_1} \cup N_{y_j}$ must have a nonempty intersection with another N_{y_i} in Z ($i \neq 1, j$) and, again, the number of solutions must be constant on the union $N_{y_1} \cup N_{y_j} \cup N_{y_i}$. By repeating this argument a finite number of times (until we have exhausted the covering Z), we can conclude that the number of solutions to the equation $(\partial f_i/\partial x_i)(y, x_i) = 0$ with fixed y is the same for all $y \in K$ and, therefore, each connected component of S_i projects onto K and is the graph of a C^1 function h: $K \to [-1, 1]$.

By the boundary conditions and by the hypothesis that $(\partial^2 f_i/\partial x_i^2)$ is never zero on S_i (A.3), at least one of these connected components consists of local maxima of the function $f_i(y, x_i)$ as a function of x_i parametrised by y (that is, by the boundary conditions (A.2) for each $y \in K$ there exists at least one point $(y, x_i) \in S_i$ such that $(\partial^2 f_i/\partial x_i^2)(y, x) < 0$, and by (A.3) $\partial^2 f_i/\partial x_i^2$ must have constant sign on each connected component of S_i).

Finally, let h_i: $K \to [-1, 1]$ be the C^1 function whose graph gives the chosen connected component of S_i. Let H_i: $C \to [-1, 1]$ be defined by $H_i(y, x_i) = h_i(y)$

and define $H: C \to C$ by $H = (H_1, \ldots, H_n)$. The function H is clearly continuous and, by Brouwer's fixed-point theorem, there exists a fixed point $x^\circ = H(x^\circ)$. Then, $x_i^\circ = H_i(x^\circ) = h_i(x_{-1}^\circ)$, where $x_{-i}^\circ = (x_1^\circ, \ldots, x_{i-1}^\circ, x_{i+1}^\circ, \ldots, x_n^\circ)$, which means that $(\partial f_i/\partial x_i)(x^\circ) = 0$ and $(\partial^2 f_i/\partial x_i^2)(x^\circ) < 0$. Thus, x° is a local Nash equilibrium. Q.E.D.

The proposition can now be shown to be an application of the above lemma using exactly the same argument which was used in the proof of Proposition 2 in Bonanno-Zeeman (1985, p. 282). We first restrict ourselves to the compact set $A = A_1 \times \cdots \times A_n$, $A_i = [c_i, p_i^*]$ and replace the profit function P_i with the function $\bar{P}_i$ obtained by prolonging the tangent at p_i' instead of going along the price axis (see Figure 5). Finally, we define $f_i(x) = \bar{P}_i(T(x))$, where T is the orientation-preserving affine map which takes C onto A. The above lemma then applies. It only remains to show that (9) implies (A.3). It is easy to check that (9) is equivalent to:

$$\text{for all } i \text{ and } p, \quad P_i(p) > 0 \quad \text{and} \quad (\partial P_i/\partial p_i)(p) = 0 \quad \text{implies} \quad (\partial^2 P_i/\partial p_i^2)(p) \neq 0.$$

In fact, $P_i(p) > 0$ and $(\partial P_i/\partial p_i)(p) = 0$ is equivalent to

$$(\partial D_i/\partial p_i)(p) = -D_i(p)/(p_i - c_i) \tag{A.4}$$

and, given (A.4), $(\partial^2 P_i/\partial p_i^2)(p) \neq 0$ is equivalent to (9). Q.E.D.

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