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Set-Theoretic Equivalence of Extensive-Form Games

G. Bonanno[1]

Abstract: A new game-form, the *set-theoretic form*, is introduced and it is shown that a set-theoretic form can be associated with every extensive form. The map from extensive forms to set-theoretic forms is not one-to-one and this fact is used to define a notion of equivalence for extensive games. A transformation for extensive forms is then defined, called the *interchange of contiguous simultaneous moves*, and it is shown that it is possible to move from one game to any other game in the same equivalence class by using this transformation a finite number of times and without ever leaving the equivalence class. This transformation is a generalization of Thompson's "interchange of decision nodes". Thus given an extensive game G there is a different extensive game G' that is equivalent to G if and only if there are moves in G that are simultaneous and the difference between G and G' lies exactly in the fact that (some of) these moves are taken in a different temporal order in the two games.

1 Introduction

In the literature on non-cooperative games two different ways of describing interactive situations have been suggested: the *normal form* (and its relatives: semi-reduced normal form, reduced normal form, standard form) and the *extensive form* (recently Greenberg [5] introduced a new modeling tool: the inducement correspondence). The extensive form gives richer descriptions than the normal form since it allows one to specify such details as the temporal order in which players move, the information available to a player when it is her turn to move, etc. Sometimes, however, the extensive form forces the modeler to incorporate "too many" details. Consider, for example, the problem of describing a situation where two players have to choose an action in ignorance of the other player's choice. The extensive form requires that the temporal order in which choices are made be specified. An example is the Battle of the Sexes. There are two ways of representing it as an extensive game and these are shown in Figure 1.

These are two different descriptions: in game G_1 player 1 knows that player 2 will make her choice after him, while in game G_2 player 1, when it is his turn to move, knows that player 2 has already chosen (although he does not know what she chose). Since G_1 and G_2 describe different situations, it is conceivable that a solution concept may prescribe different solutions for the two games. Indeed there is a refinement of Nash equilibrium, recently proposed by Amershi et al. [1], according to

[1] Giacomo Bonanno, Department of Economics, University of California, Davis, CA 95616, USA.

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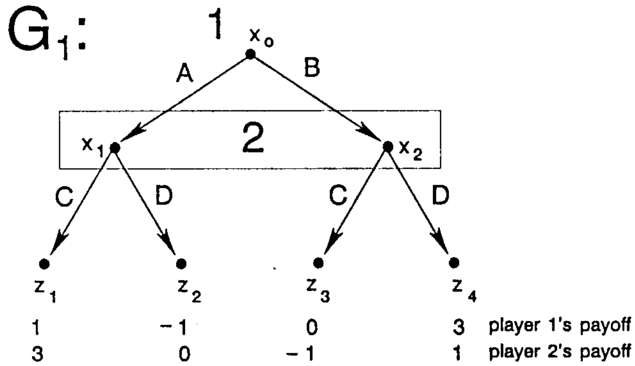

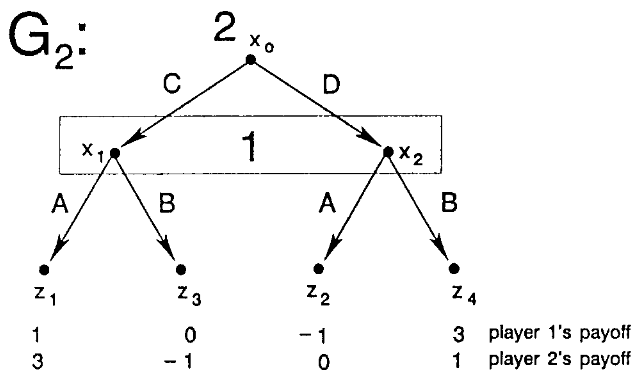

Fig. 1

which G_1 has a unique solution where player 1's payoff is 3 and player 2's payoff is 1, while G_2 has a unique solution where player 1's payoff is 1 and player 2's payoff is 3.

One can react to this fact in two different ways, depending on whether one is arguing in the "description mode" or in the "solution mode".

Somebody who argues in the description mode might suggest that if one doesn't want to be forced to specify such details as the order in which simultaneous actions are taken, then one can always use the normal form rather than the extensive form as a modeling tool. The problem with this suggestion is that it is not satisfactory to have a theory of games where sometimes situations are described using a certain tool (the normal form) and sometimes using a different tool (the extensive form): a coherent theory should make use of the same language to model every situation. But then, if one chooses the normal form as a modeling tool, one loses the ability to specify the temporal order of moves in *every* situation, including those where moves are *not* simultaneous. Consider the following example: player 1 is first asked to choose whether or not to end the game by taking action Y, which gives a payoff of 2 to both players. If player 1 decides not to do so, a simultaneous Battle of the Sexes follows. One way of representing this as an extensive form is shown as game G_4 in Figure 2.

The extensive form requires us to specify who moves first after player 1 has chosen action N, while the normal form forces us to give up the sequentiality of actions A, B, C and D with respect to action N.

Those who argue in the solution mode, on the other hand, might suggest that the details over which the two extensive games of Figure 1 differ are strategically irrelevant, in other words rational players would not make their choices depend on those details (this, of course, implies a notion of rationality that must be different from the one implicit in the solution concept put forward by Amershi et al. [1]). This objection raises the following question: when is it that two extensive games are "essentially the same", in the sense that rational players would make the "same" choices in the two games? It seems that the only satisfactory way of answering this question is to start from an extensive-form solution concept (or, even better, an explicit definition of rationality) and then define two extensive games to be strategically equivalent if and only if they have the same solution(s). Kohlberg and Mertens [6]

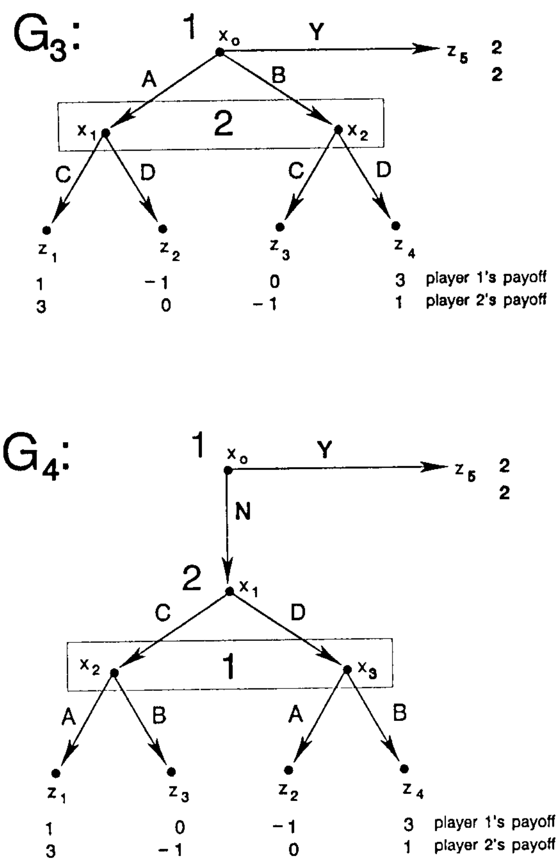

Fig. 2

in their very influential contribution did not pursue this approach but suggested a different one. They went back to a result of Thompson [9] according to which two extensive games have the same (reduced) normal form if and only if one can be obtained from the other by applying one or more of four well-defined transformations. Kohlberg and Mertens maintained that these transformations *ought to be* considered "irrelevant" by rational players and suggested a solution concept defined on the reduced normal form. Obviously, their solution concept is compelling, as a solution concept for extensive games, only if one accepts their *axiom* that those transformations are indeed "irrelevant".

This paper is concerned not with rationality, or strategic considerations, but with the descriptive component of the theory of non-cooperative games. It is therefore in the same spirit as the papers by Dalkey [3], Elmes and Reny [4], Krentel et al. [7], and Thompson [9].

We start by defining a new game-form, called *set-theoretic form*. We show that a set-theoretic form can be associated with every extensive form. However, the map from extensive forms to set-theoretic forms is not one-to-one and we use this fact to define a notion of equivalence of extensive forms. It is worth stressing that we are not suggesting a notion of strategic equivalence but rather a notion of "descriptive" equivalence.

Next we describe a transformation, called the *interchange of contiguous simultaneous moves*, and show that it is possible to move from one extensive game to an equivalent one by using this transformation a finite number of times and without ever leaving the equivalence class. This transformation is a generalization of Thompson's "interchange of decision nodes". Thus given an extensive game G there is a different extensive game G' which is (set-theoretically) equivalent to G if and only if there are moves in G that are simultaneous and the difference between G and G' lies exactly in the fact that (some of) these moves are taken in a different temporal order in the two games (for example, it will be shown that the two extensive games of Figure 1 are set-theoretically equivalent). Thus the game-form suggested in this paper has all the richness of the extensive form, excluding only the specification of the temporal order in which simultaneous moves are made.

2 Definition of Game in Set-Theoretic Form

Throughout the paper we shall restrict attention to finite games without chance moves.

Definition 2.1. A finite (non-cooperative) *game in set-theoretic form* (without chance moves) is a $(3n+2)$-tuple

$$\langle N, \Omega, \{\pi_i\}_{i\in N}, \{\mathscr{A}_i\}_{i\in N}, \{\Sigma_i\}_{i\in N}\rangle$$

where

$N = \{1, 2, \ldots, n\}$ is a finite set of *players*;

Ω is a finite set of *outcomes*;

π_i: $\Omega \to \Re$ is player i's *payoff function* ($i \in N$);

$\mathscr{A}_i$ is a collection of non-empty subsets of Ω; an element A of $\mathscr{A}_i$ is called an *action* of player i;

Σ_i is a partition of $\mathscr{A}_i$; an element $\mathscr{S}$ of Σ_i is called a *situation* for player i.

The intuition behind the above definition is that taking an action means narrowing down the set of possible outcomes. Thus an action can be thought of as a subset of the set of initially possible outcomes: when player i takes action A, the result is that the outcome which will eventually obtain is restricted to the subset A of Ω. (Note that we have ruled out impossible actions, that is, actions A such that $A = \emptyset$). A situation for player i is simply a collection of actions among which the player has to choose.

In order to make a game in set-theoretic form "playable", that is, in order to specify how players would play such a game, more structure needs to be added. However, since our concern in this paper is merely with equivalence of extensive games, the simple structure of definition 2.1 will suffice (and our results will correspondingly be stronger).

We now show how to associate with every game in extensive form a game in set-theoretic form by mapping terminal nodes into outcomes, choices into actions and information sets into situations. This map, however, is not one-to-one and we shall use this fact to define a notion of equivalence for extensive games (section 3).

We shall adopt the definition of (finite) extensive game given by Selten [8] and restrict attention to extensive games without chance moves. Given an extensive game G (not necessarily with perfect recall), let Z be its set of terminal nodes and let $\Omega = Z$, that is, let the set of outcomes coincide with the set of terminal nodes. Let N coincide with the set of players in G and let π_i coincide with $h_i: Z \to \Re$, the latter being player i's payoff function in G. Recall that a choice c of player i at one of his information sets in G is identified with a set of arcs (or arrows), one for each node in the information set. With each choice c in G we associate the set A of terminal nodes that can be reached by plays that have an arc in common with c. We call A the action corresponding to choice c. More precisely, let E be the set of arcs in the game tree. For every $e \in E$, let $\lambda(e) \subseteq Z$ be the set of terminal nodes that can be reached by plays that contain arc e. If c is a choice (hence a set of arcs), define $\mu(c) = \bigcup_{e \in c} \lambda(e)$. Then $\mu(c)$ is the action corresponding to choice c. Thus $\mathscr{A}_i = \{\mu(c)\}_{c \in C_i}$ where C_i is the set of choices of player i in G. Finally, we associate with every information set u of player i in G a situation $\mathscr{S}(u)$ as follows: if $c_1, \ldots, c_m$ are player i's choices at u, then $\mathscr{S}(u) = \{\mu(c_1), \ldots, \mu(c_m)\}$. It is clear that the object so constructed is a game in set-theoretic form as defined above. (Note that, for an extensive game without trivial moves, the function $u \mapsto \mathscr{S}(u)$ from information sets to situations is one-to-one: see Bonanno [2], where it is also shown that such notions as perfect recall, perfect information, simultaneity, etc. can be given a very appealing formulation in the set-theoretic form).

Example 2.1. The extensive games G_1 and G_2 of Figure 1 are both mapped into the following game in set-theoretic form:

$$N=\{1,2\},\quad \Omega=\{z_1,z_2,z_3,z_4\},$$
$$\pi_1(z_1)=1,\quad \pi_1(z_2)=-1,\quad \pi_1(z_3)=0,\quad \pi_1(z_4)=3,$$
$$\pi_2(z_1)=3,\quad \pi_2(z_2)=0,\quad \pi_2(z_3)=-1,\quad \pi_2(z_4)=1,$$
$$\mathscr{A}_1=\{A\equiv\{z_1,z_2\},B\equiv\{z_3,z_4\}\},\quad \Sigma_1=\{\mathscr{A}_1\}$$
$$\mathscr{A}_2=\{C\equiv\{z_1,z_3\},D\equiv\{z_2,z_4\}\},\quad \Sigma_2=\{\mathscr{A}_2\}.$$

Example 2.2. Game G_3 of Figure 2 is mapped into the following game in set-theoretic form:

$$N=\{1,2\},\quad \Omega=\{z_1,z_2,z_3,z_4,z_5\},$$
$$\pi_1(z_1)=1,\quad \pi_1(z_2)=-1,\quad \pi_1(z_3)=0,\quad \pi_1(z_4)=3,\quad \pi_1(z_5)=2,$$
$$\pi_2(z_1)=3,\quad \pi_2(z_2)=0,\quad \pi_2(z_3)=-1,\quad \pi_2(z_4)=1,\quad \pi_2(z_5)=2,$$
$$\mathscr{A}_1=\{Y\equiv\{z_5\},A\equiv\{z_1,z_2\},B\equiv\{z_3,z_4\}\},\quad \Sigma_1=\{\mathscr{A}_1\}$$
$$\mathscr{A}_2=\{C\equiv\{z_1,z_3\},D\equiv\{z_2,z_4\}\},\quad \Sigma_2=\{\mathscr{A}_2\},$$

while game G_4 of Figure 2 is mapped into the following game in set-theoretic form:

$$N,\ \Omega,\ \pi_1,\ \pi_2,\ \mathscr{A}_2 \text{ and } \Sigma_2 \text{ as above},$$
$$\mathscr{A}_1=\{Y\equiv\{z_5\},N\equiv\{z_1,z_2,z_3,z_4\},A\equiv\{z_1,z_2\},B\equiv\{z_3,z_4\}\},$$
$$\Sigma_1=\{\{Y,N\},\{A,B\}\}.$$

Thus, Figure 2 shows an example of two games that have the same reduced normal form and yet have different set-theoretic forms.

3 Equivalence of Extensive Games

From now on we shall restrict attention to extensive games without trivial moves, by which we mean that every player at every information set has at least two choices. Recall, however, that we are not limiting ourselves to extensive games with perfect recall.

Definition 3.1. Two extensive games G and G' are *equivalent* if they are both mapped into the same game in set-theoretic form (up to renaming). [It is clear that the following binary relation R on the set of extensive games: "GRG' if the set-theoretic form obtained from G is the same as that obtained from G' by applying the procedure explained in section 2" is an equivalence relation, that is, it is reflexive, symmetric and transitive.]

Thus, for example, the two extensive games of Figure 1 are equivalent (cf. example 2.1), while the two extensive games of Figure 2 are not (cf. example 2.2).

It is clear that if G and G' are equivalent, then they have:

(i) the same set of players,
(ii) the same set of terminal nodes (hence the same number of plays),
(iii) the same payoff functions,
(iv) the same number of information sets for each player.

They may, however, have a different number of decision nodes and arcs, as the equivalent games of Figure 3 show. (From now on, for simplicity, we shall omit payoffs in the Figures.)

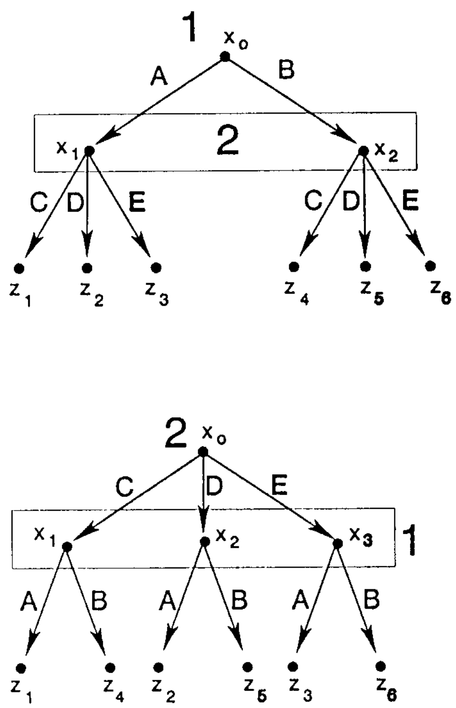

Fig. 3

Lemma 3.1. Let G and G′ be equivalent extensive games. Then for every terminal node z, the play to z in G has associated with it the same set of actions as the play to z in G′ (and hence the same number of arcs).

Proof. In this and the following proofs we shall make extensive use of the functions λ und μ defined in section 2. It may be worth recalling those definitions. Given an extensive game G where Z is the set of terminal nodes, E the set of arcs and C the set of choices,

$$\lambda : E \to \mathscr{P}(Z) \tag{1}$$

(where $\mathscr{P}(Z)$ is the set of subsets of Z) is the function that associates with every arc e the set of terminal nodes that can be reached by plays that contain e, while

$$\mu : C \to \mathscr{P}(Z) \tag{2}$$

associates with every choice c the action $\mu(c) = \bigcup_{e \in c} \lambda(e)$. Finally, we shall define a new function:

$$\xi : E \to \mathscr{P}(Z) \tag{3}$$

as follows: $\xi(e) = \mu(c(e))$ where $c(e)$ is the unique choice to which arc e belongs. (When we consider an extensive game G' different from G, we shall denote the corresponding functions by λ', μ' and ξ').

Now, let $\langle N, \Omega, \{\pi_i\}_{i \in N}, \{\mathscr{A}_i\}_{i \in N}, \{\Sigma_i\}_{i \in N}\rangle$ be the game in set-theoretic form associated with G and G'. Fix $z \in Z$. Let P_z be the set of arcs that belong to the play to z in G and P'_z be the set of arcs that belong to the play to z in G' (we distinguish between the play to z, denoted by p_z, which is an ordered sequence of arcs, and P_z which is the (unordered) set containing those arcs). By definition of the function ξ, $\xi(P_z)$ is the set of actions associated, in G, with the arcs that belong to P_z. Similarly $\xi'(P'_z)$ is the set of actions associated, in G', with the arcs that belong to P'_z. We want to show that $\xi(P_z) = \xi'(P'_z)$. Suppose not, that is, suppose there is an action A that belongs to $\xi'(P'_z)$ but not to $\xi(P_z)$. Since $A \in \xi'(P'_z)$, $z \in A$. Since G and G' are equivalent, there is a choice c in G whose corresponding action is A, that is, $A = \mu(c)$. Since $z \in A$, there is an arc $e \in c$ such that $z \in \lambda(e)$. Hence there is, in G, a play to z that goes through e. If e does not belong to P_z then we contradict the fact that the play to z is unique. Hence $e \in P_z$. But $e \in c$ implies that $\xi(e) = \mu(c)$. Hence $A \in \xi(P_z)$, a contradiction. □

It follows from lemma 3.1 that if G is equivalent to G' and $G \neq G'$, then the difference must lie in the order in which actions are taken along one or more plays.

4 The Interchange of Contiguous Simultaneous Moves

We now describe a transformation, called the *interchange of contiguous simultaneous moves*, that leads from an extensive game G to an equivalent extensive game $G' \neq G$. This transformation is illustrated in Figure 4 and is a generalization of the transformation "interchange of decision nodes" introduced by Thompson [9].

Let G be an extensive game and let x be a node that belongs to information set u of player i (note that u may contain other decision nodes besides x). Let $e_1 = (x, y_1), \ldots, e_m = (x, y_m)$ be the arcs incident out of x. Suppose that decision nodes $y_1, \ldots, y_m$ all belong to the same information set v of player j (not necessarily $j \neq i$; note also that v may contain other decision nodes too). Let r be the number of choices of player j at his information set v. Then there are r arcs incident out of each y_k. Denote them by $(y_k, t_{k1}), (y_k, t_{k2}), \ldots, (y_k, t_{kr})$ $(k = 1, \ldots, m)$. We say that i's moves at x and j's moves at the nodes $y_1, \ldots, y_m$ are *contiguous and simultaneous*. These moves can be *interchanged* as follows. Assign node x to player j and replace

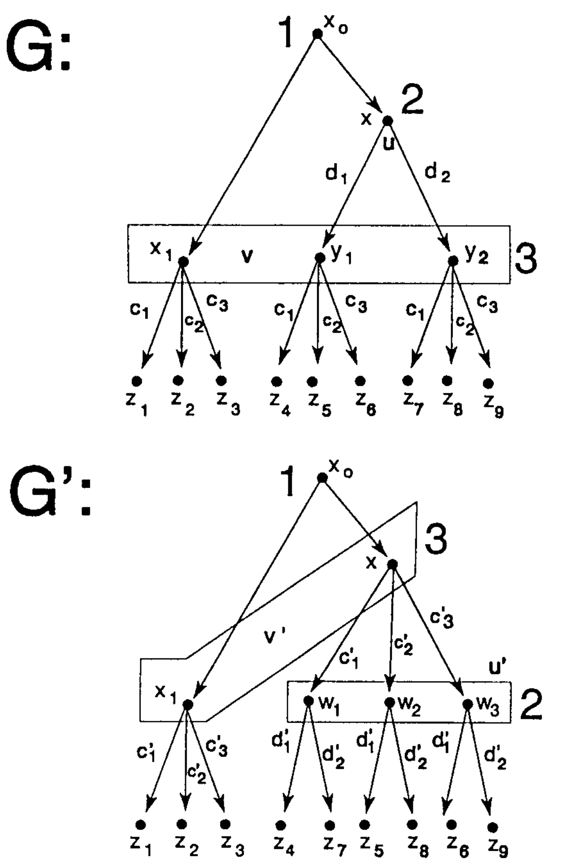

Fig. 4

her information set v with $v' = (v \setminus \{y_1, \ldots, y_m\}) \cup \{x\}$. Draw r nodes $w_1, \ldots, w_r$ and r arcs $f_1 = (x, w_1), \ldots, f_r = (x, w_r)$ incident out of x. Assign the set of decision nodes $\{w_1, \ldots, w_r\}$ to player i and replace his information set u with $u' = (u \setminus \{x\}) \cup \{w_1, \ldots, w_r\}$. Let c_q be a choice of player j at information set v $(q = 1, \ldots, r)$ and let arcs $(y_1, t_{1\alpha(q,1)}), (y_2, t_{2\alpha(q,2)}), \ldots, (y_m, t_{m\alpha(q,m)})$ belong to c_q, where $\alpha(q, 1), \alpha(q, 2), \ldots, \alpha(q, m)$ are integers between 1 and r. Draw m arcs out of decision node w_q as follows: $(w_q, t_{1\alpha(q,1)}), (w_q, t_{2\alpha(q,2)}), \ldots, (w_q, t_{m\alpha(q,m)})$. Replace choice c_q of player j with $c'_q = (c_q \setminus \{(y_1, t_{1\alpha(q,1)}), (y_2, t_{2\alpha(q,2)}) \ldots, (y_m, t_{m\alpha(q,m)})\}) \cup \{(x, w_q)\}$. Finally, let arc (x, y_k) belong to choice d_k of player i $(k = 1, \ldots, m)$. Replace d_k with $d'_k = (d_k \setminus \{(x, y_k)\}) \cup \{(w_1, t_{k1}), (w_2, t_{k2}), \ldots, (w_r, t_{kr})\}$. That the game thus obtained is equivalent to the initial game is shown in proposition 4.1.

Figure 4 illustrates this transformation. In this case we have: $i = 2$; $m = 2$; $j = 3$; $r = 3$; $t_{11} = z_4$, $t_{12} = z_5$, $t_{13} = z_6$, $t_{21} = z_7$, $t_{22} = z_8$, $t_{23} = z_9$; $c_1 = \{(x_1, z_1), (y_1, z_4), (y_2, z_7)\}$, $c_2 = \{(x_1, z_2), (y_1, z_5), (y_2, z_8)\}$, $c_3 = \{(x_1, z_3), (y_1, z_6), (y_2, z_9)\}$, so that $\alpha(1,1) = \alpha(1, 2) = 1$, $\alpha(2, 1) = \alpha(2, 2) = 2$, $\alpha(3, 1) = \alpha(3, 2) = 3$; $d_1 = \{(x, y_1)\}$, $d_2 = \{(x, y_2)\}$, etc.

Proposition 4.1. Let G' be an extensive game obtained from G by applying (once) the transformation of interchange of contiguous simultaneous moves (from now on ICSM) described above. Then G' is equivalent to G.

Proof. The following are the only differences between G and G':

(i) decision node x belongs to player i's information set u in G and to player j's information set v' in G';
(ii) decision nodes $y_1, \ldots, y_m$ in G (which belong to player j's information set v) are replaced by decision nodes $w_1, \ldots, w_r$ in G' (which belong to information set u' of player i);
(iii) choice d_k of player i in G, which contains arc (x, y_k), is replaced in G' by choice $d'_k = (d_k \setminus \{(x, y_k)\}) \cup \{(w_1, t_{k1}), \ldots, (w_r, t_{kr})\}$, for each $k = 1, \ldots, m$;
(iv) choice c_q of player j in G, which contains arcs $(y_1, t_{1\alpha(q,1)}), (y_2, t_{2\alpha(q,2)}), \ldots, (y_m, t_{m\alpha(q,m)})$ [where each $\alpha(q, k)$ is an integer between 1 and r], is replaced in G' by $c'_q = (c_q \setminus \{(y_1, t_{1\alpha(q,1)}), (y_2, t_{2\alpha(q,2)}), \ldots, (y_m, t_{m\alpha(q,m)})\}) \cup \{(x, w_q)\}$.

G and G' coincide in everything else. Thus we only need to show that if, in game G, $\mu(d_k) = A \in \mathscr{A}_i$, then $\mu'(d'_k) = A$ in G', for every $k = 1, \ldots, m$; and, similarly, if, in game G, $\mu(c_q) = B \in \mathscr{A}_j$, then $\mu'(c'_q) = B$ in G'. It is clear that the set of terminal nodes reached by d_k starting from all the decision nodes in u different from x is the same as the set of terminal nodes reached by d'_k starting from all the decision nodes in u' different from $w_1, w_2, \ldots, w_r$. Hence we only need to restrict attention to node x in u and nodes $w_1, w_2, \ldots, w_r$ in u'. Now, in game G, d_k contains arc (x, y_k) which reaches the set of nodes $\{t_{k1}, t_{k2}, \ldots, t_{kr}\}$, while in G' d'_k contains arcs (w_1, t_{k1}), $(w_2, t_{k2}), \ldots, (w_r, t_{kr})$ which also reach the set of nodes $\{t_{k1}, t_{k2}, \ldots, t_{kr}\}$. Hence $\mu(d_k) = \mu'(d'_k)$ for every $k = 1, \ldots, m$. Similarly, for every $q = 1, \ldots, r$, in game G c_q contains arcs $(y_1, t_{1\alpha(q,1)}), \ldots, (y_m, t_{m\alpha(q,m)})$, while in game G' c'_q contains arc

(x, w_q) and in both cases the set of nodes $\{t_{1\alpha(q,1)}, \ldots, t_{m\alpha(q,m)}\}$ is reached. Hence $\mu(c_q) = \mu'(c'_q)$. □

The rest of the paper is aimed at showing the sufficiency of this transformation.

5 Games with Perfect Information and Simultaneous Games

It is clear that in order to be able to apply the transformation ICSM to a given extensive game G, it is necessary that G satisfy the following property: there exists a node x and an information set v such that: (a) $x \notin v$ and x precedes v, (b) all the plays through x cross v. [Note that this implies that for every node y that lies on a play through x and is between x and v it is also true that every play through y crosses v: it is a consequence of the fact that in a tree for every two nodes there is at most one path connecting them]. Let us call the negation of this property, property β.

Definition 5.1. We say that an extensive game G *satisfies property* β if for every decision node x and for every information set v which is preceded by x (that is, v contains a node that is a successor of x), there exists a play through x that does *not* cross v.

For example, an extensive game of perfect information without trivial moves satisfies property β.

Proposition 5.1. Let G be an extensive game that satisfies property β and let G' be an extensive game that is (set-theoretically) equivalent to G. Then $G' = G$. In other words, G is the only member of its equivalence class.

Proof. If G is an extensive game and u an information set of G, we define the *span of* u, denoted $sp(u)$, to be the set of terminal nodes that can be reached starting from nodes in u. Hence

$$sp(u) = \bigcup_{A \in \mathscr{S}(u)} A$$

(recall that $\mathscr{S}(u)$ is the situation corresponding to information set u).

Let G be an extensive game that satisfies property β. Let $u = \{x_0\}$, where x_0 is the root of G. Then $sp(u) = Z$ and, for every information set $v \neq u$, $sp(v)$ is a proper subset of Z (recall that we are only considering extensive games without trivial moves). Let G' be an extensive game that is equivalent to G. Let u' be the information set in G' that corresponds to u [that is, $\mathscr{S}(u) = \mathscr{S}'(u')$]. We want to show that $u' = \{x'_0\}$, where x'_0 is the root of G'. Suppose not, that is, suppose $u' \neq \{x'_0\} \equiv v'$. Then $sp(v') = Z$. Let v be the information set in G corresponding to v'. Then $v \neq u$ by our supposition (recall that in extensive games without trivial moves the function $u \mapsto \mathscr{S}(u)$ is one-to-one) and $sp(v) = sp(v') = Z$, contradicting the hypothesis that G satisfies property β. Hence G and G' have the same player and the same situation at the root. Thus for every action $A \in \mathscr{S}(\{x_0\})$ there is a unique arc e in G (incident out of the root) corresponding to action A (that is, $\xi(e) \equiv \mu(\{e\}) = A$) and a unique arc e' in G' (incident out of the root) corresponding to action A (that is, $\xi'(e') \equiv \mu'(\{e'\}) = A$). Now fix an arbitrary $A \in \mathscr{S}(\{x_0\})$ and let $e = (x_0, y)$ be the arc such that $\xi(e) = A$. Let v be the information set to which y belongs (if y is a terminal node, choose another action and another arc out of x_0; if all the arcs out of x_0 end at a terminal node, then clearly $G = G'$). Let v' be the information set in G' that corresponds to v. Let $e' = (x_0', y')$ be the arc in G' such that $\xi'(e') = A$. We want to show that $y' \in v'$. Suppose not. Let $y' \in w'$, with $w' \neq v'$. Then in G' all the plays with A as first action cross w'. Let w be the information set in G that corresponds to w' [that is, $\mathscr{S}(w) = \mathscr{S}'(w')$, hence $sp(w) = sp(w')$]. By our supposition $w \neq v$. By lemma 3.1 also in G all the plays with A as first action must cross w (the plays with A as first action are in one-to-one correspondence with the elements of A). But since $w \neq v$, it follows that $y \notin w$ and y precedes w. Furthermore, in G all plays with A as first action go through node y. Hence all the plays that go through node y cross w, contradicting the hypothesis that G satisfies property β.

Since action $A \in \mathscr{S}(\{x_0\})$ was chosen arbitrarily, we have shown that for every $z \in Z$, the first two information sets crossed by the plays to z in G and G', and the order in which they are crossed, are the same in the two games. Again, fix an arbitrary $A \in \mathscr{S}(\{x_0\})$ and let (x_0, y) be the corresponding arc in G. Let v be the information set to which y belongs. Fix an arbitrary arc $e = (y, x)$ and let $B = \xi(e)$ be the corresponding action and let w be the information set to which x belongs (as before, if x is a terminal node, choose another arc out of y; if all arcs out of y end at terminal nodes, choose another path of length 2 out of the root; if all the paths of length 2 out of the root end at terminal nodes, then $G = G'$). Let $e' = (x_0', y')$ be the arc corresponding to A in G' (that is, $\xi'(e') = A$) and $e'' = (y', x')$ be the arc corresponding to action B, that is, $\xi'(e'') = B$ [we showed above that $y' \in v'$, where v' is the information set in G' corresponding to v, that is, $\mathscr{S}(v) = \mathscr{S}'(v')$]. Let w' be the information set in G' corresponding to w. We want to show that $x' \in w'$. Suppose not, that is, suppose $x' \in t' \neq w'$. Then in G' all the plays with A and B as the first two actions cross t'. Let t be the information set in G that corresponds to t'. By our supposition $t \neq w$ and therefore $x \notin t$. By lemma 3.1 also in G all the plays with A and B as first actions must cross t (these plays are in one-to-one correspondence with the elements of $A \cap B$). But those plays go through node x, contradicting the hypothesis that G satisfies property β. This argument can be repeated for every decision node to show that for every $z \in Z$ the play to z in G and the play to z in G' have associated with them the same *ordered* set of actions. Hence $G = G'$. □

Corollary 5.2. If G is an extensive game with perfect information (and without trivial moves), then G is the only member of its equivalence class.

Corollary 5.3. Let G be an extensive game. A necessary and sufficient condition for there to be an extensive game G′ that is equivalent but not identical to G is that there be a node x and an information set v such that: (a) x∉v and x precedes v, (b) all the plays through x cross v.

Proof. Necessity is a corollary of proposition 5.1, sufficiency is a corollary of proposition 4.1. □

Define an extensive game G to be *simultaneous* if every play crosses all the information sets. (Thus, a simultaneous game without trivial moves has perfect recall if and only if each player has exactly one information set.)

Note, therefore, that if G is a game of length two (the length of a game is defined as the length of its longest play), G satisfies property β if and only if G is not simultaneous. The same is not true for games of length three or more. Thus another corollary of proposition 5.1 is that if G is a game that is not simultaneous and is of length 2, then G is the only game in its equivalence class. For example, there is no game which is equivalent to game G_3 of Figure 2, apart from G_3 itself.

Proposition 5.2. Let G be a simultaneous extensive game (without trivial moves). Then:

- *(i) If G' is equivalent to G, then G' is also simultaneous;*
- *(ii) If each player has exactly one information set (i.e. if G has perfect recall) then the equivalence class of G contains*

  $$\sum_{i=1}^{n} [(n-1)!]^{m(i)}$$

  games, where $m(i)$ is the number of choices of player i at his information set and n is the number of players. (The same formula applies to the case where one or more players have more than one information set, provided it is re-interpreted as follows. Let $r(i)$ be the number of information sets of player i $(i = 1, 2, \ldots, n)$ and define $n' = r(1) + r(2) + \ldots + r(n)$, that is, treat the same player at different information sets as different players. Then replace n with n' in the above formula.)

Proof.

- (i) Let G be a simultaneous extensive game and G' be equivalent to G. Since for every extensive game (without trivial moves) the map from information sets to situations is one-to-one (see Bonanno [2]) and both G and G' have the same set of situations for every player, it follows that for every player i there is a one-to-one map between his information sets in G and his information sets in G'. Since in extensive games with no trivial moves every action belongs to one and only one situation (see Bonanno [2]), the fact that G' is simultaneous follows from lemma 3.1.
- (ii) Let G be a simultaneous game and assume that each player has exactly one information set. In virtue of lemma 3.1 if we vary the order in which actions are taken along any given play, with the constraint that all the first actions belong to the same situation, we obtain a game which is equivalent to G. Thus to obtain all the games in the equivalence class of G we can proceed as follows. Assign a player to the root. There are n possible ways of doing this. Let i be the root player and $m(i)$ be the number of his actions. For each action of player i choose a permutation of the remaining $(n-1)$ players. Thus the total number of possible games where player i is at the root is $[(n-1)!]^{m(i)}$. □

Figure 1 shows the equivalence class of a two-player simultaneous game where each player has one information set and two choices, while Figure 5 shows the equivalence class of a three-player simultaneous game where each player has one information set and two choices.

So far we have dealt with the two extreme cases of games of perfect information and of simultaneous games. The next section deals with general (finite) extensive games (with no trivial moves and no chance moves).

6 General Extensive Games

The following proposition contains the main result of this paper. We recall once more that we only consider extensive games without trivial moves.

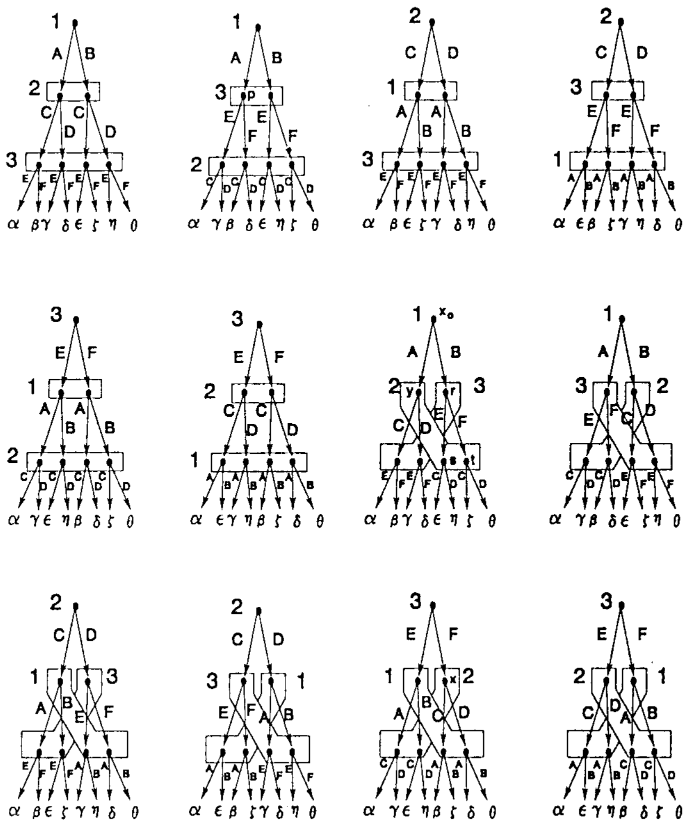

Fig. 5

Proposition 6.1. Let G_0 and G_0' be two equivalent extensive games. Then there is a finite sequence of equivalent games $\langle G_1, \ldots, G_m \rangle$ such that:

- *(i)* $G_1 = G_0$;
- *(ii)* $G_m = G_0'$;
- (iii) for every $k=2, \ldots, m$, *G_k is obtained from G_{k-1} by applying the transformation of interchange of contiguous simultaneous moves.*

We shall first prove the following lemma. Define an information set to be *maximal* if every play crosses it (for example, the root is a maximal information set).

Lemma 6.2. Let G_0 and G_0' be two (set-theoretically) equivalent extensive games. Let m be the number of maximal information sets in G_0. Then also G_0' has exactly m maximal information sets. Furthermore, there is an ordering $\langle \mathcal{S}_1, \ldots, \mathcal{S}_m \rangle$ of the situations corresponding to these information sets such that – with a finite number of applications of the transformation ICSM – it is possible to transform G_0 into an equivalent game G where all the paths of length m from the root cross $\mathcal{S}_1, \ldots, \mathcal{S}_m$ in this order; similarly – with a finite number of applications of ICSM – it is possible to transform G_0' into an equivalent game G' where all the paths of length m from the root cross $\mathcal{S}_1, \ldots, \mathcal{S}_m$ in this order. Thus G and G' are equivalent (by proposition 4.1) and the m-truncation of G coincides with the m-truncation of G'. (By the m-truncation of a game ($m \geq 0$) we mean what is left of the game by considering only paths of length m from the root and replacing what comes after a node x which is at the end of such a path with $\theta(x)$, where $\theta(x)$ denotes the set of terminal nodes that can be reached starting from node x).

Proof. We shall describe an algorithm that transforms G_0 into G and G_0' into G'. This algorithm will then be illustrated with an example based on Figure 5. First of all, note that, by lemma 3.1, if u is a maximal information set of G_0 then the corresponding information set in G_0' is maximal in G_0'. Thus G_0 and G_0' have the same number of maximal information sets. Thus if $m=1$, it follows that the root is the only maximal information set in both games and by the argument used in the proof of proposition 5.1 both games have the same player at the root and the same situation. Suppose therefore that $m>1$.

STEP 1. Let player i be the player at the root of G_0 and let $u=\{x_0\}$. Let u' be the corresponding information set of player i in G_0'. If $u'=\{x_0'\}$ go to step 2. If $u' \neq \{x_0'\}$, then $x_0' \notin u'$ because no play can cross an information set more than once. Since u' is a maximal information set in G_0', every play crosses u'. For every $z \in Z$, let $y'(z)$ be the immediate predecessor of u' on the play to z. Consider one which is farthest from x_0': call it $y'(z_0)$. Apply the transformation lCSM to $y'(z_0)$. Let G_0'' be the new game thus obtained. By proposition 4.1, G_0'' is equivalent to G_0'. Let u'' be the information set in G_0'' corresponding to u in G_0. There are three possibilities: (1) $u''=\{x_0''\}$; (2) u'' is the set of immediate successors of x_0'' (the root); (3) $u'' \neq \{x_0''\}$ and u'' is not equal to the set of immediate successors of x_0''. In case (1) go to step 2. In case (2) apply ICSM to x_0'' and then go to step 2. In case (3) repeat the above procedure until a game is obtained that is such that the information set of player i corresponding to u consists of the immediate successors of the root, then apply ICSM to the root and go to step 2. Given the finiteness of G_0', this can be done in a finite number of steps.

STEP 2. [At the end of step 1 we have transformed G_0' into an equivalent game that has the same player and the same situation at the root as game G_0]. Consider now game G_0. Fix an arc (x_0, x) incident out of the root and let v be the information set to which x belongs and let i be the corresponding player. There are two possibilities: (1) v is a maximal information set, (2) v is not maximal.

Suppose first that v is maximal. If every other arc incident out of x_0 is incident into v go to step 3, otherwise let (x_0, y) be an arc such that $y \notin v$. Then y precedes v and every play through y crosses v. Fix a $z \in Z$ reached by a play through y. Let $w(z)$ be the immediate predecessor of v along the play to z. Apply the transformation ICSM to $w(z)$. Repeat this procedure until a game is obtained which is such that the information set corresponding to v consists of the immediate successors of the root. We say that such an information set is *fully in second position*. Since G_0 is finite this can be done in a finite number of applications of ICSM.

If v is not maximal, fix a play through x and let t be the first maximal information set crossed by this play. Apply the procedure just described to transform game G_0 into an equivalent game where node x belongs to information set t. Then proceed as described to take t fully into second position.

STEP 3. [At the end of step 2 we have transformed G_0 into an equivalent game where all the immediate successors of the root belong to the same information set, call it v]. Go back to the game obtained at the end of step 1. Call it G_0''. Let v'' be the information set corresponding to v. If v'' consists of the immediate successors of the root, go to step 4, otherwise apply the procedure of step 2 in order to obtain a game that does have this property. If $m=2$, the proof is now complete. Suppose therefore that $m>2$.

STEP 4. Go back to the game obtained at the end of step 2 and proceed as in step 2 (with an arbitrary arc incident out of the information set corresponding to v) so as to obtain a game with a maximal information set (among the remaining ones), call it t, that consists of the immediate successors of the immediate successors of the root. We say that such a t is *fully in third position*.

STEP 5. Go back to the game obtained at the end of step 3 and proceed as in step 3 so as to obtain a game where the information set corresponding to t is fully in third position. If $m=3$ the proof is complete.

If $m>3$, continue these steps until both games have been transformed into games G and G', respectively, both having the property that every path of length m from the root crosses the m maximal information sets in the same order (that is, the corresponding ordered sequence of situations is the same in the two games). □

Example. Consider Figure 5 and number the games from left to right and from top to bottom. Let us use the above algorithm with $G_0 = G_7$ (the third game in the second row) and $G_0' = G_{11}$ (the third game in the third row).

STEP 1: first apply ICSM to node x in G_{11} and obtain G_5. Then apply ICSM to the root in G_5 and obtain G_2.

STEP 2: let us choose arc (x_0, y) in G_7 so that $i=2$ and $v=\{y, s, t\}$. Then apply ICSM to r and obtain G_1. The even-numbered steps have therefore been completed.

STEP 3: apply ICSM to node p in G_2 and obtain game G_7. Now apply ICSM to node r in G_7 and obtain G_1. The odd-numbered steps have also been completed.

Since in this example all the information sets are maximal (that is, the games are simultaneous) the two games have actually been transformed into the same game.

Define an information set u to be *maximal relative to decision node* $x \notin u$ if x precedes u and all the plays that go through x cross u.

Proof of Proposition 6.1. We describe an algorithm that transforms G_0' into G_0 by means of a finite number of applications of the transformation ICSM. Figure 6 shows an equivalence class and all the possible ways of transforming any game into any other game following the steps given below.

STEP 1. Consider first the information sets in G_0 that are maximal. Let m be their number. Apply lemma 6.2 to transform G_0 and G_0' into two games, G and G' respectively, such that: G and G' are equivalent and their m-truncations are identical, that is, for every terminal node z, the first m actions associated with the play to z in G, and the order in which they are taken, coincide with the first m actions

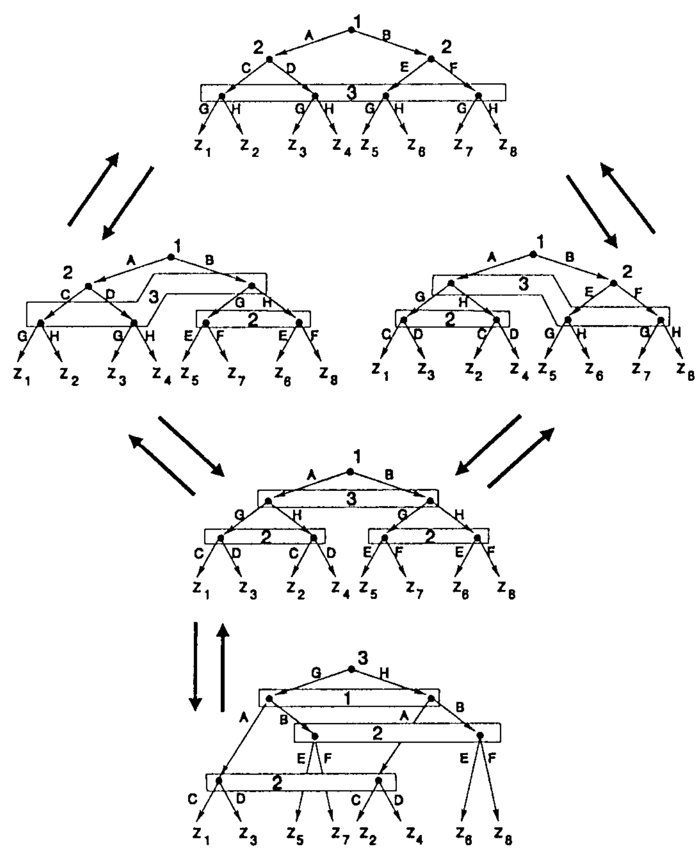

Fig. 6

associated with the play to z in G' and the order in which they are taken. Therefore if G is a game of length m (hence a simultaneous game), then so is G' and $G = G'$. Suppose therefore that G is of length greater than m.

STEP 2. Let u be the mth information set from the root in G' and u' the corresponding one in G'. There is a one-to-one correspondence between the nodes in u and those in u'. Fix $x \in u$ and let $x' \in u'$ be the corresponding node in G'. The set of terminal nodes that can be reached starting from x is equal to the set of end nodes that can be reached starting from x' and is equal to $(A_1 \cap A_2 \cap \ldots \cap A_m)$ where $A_1, A_2, \ldots, A_m$ are the first m actions associated with all the plays that go through x and x'. Suppose that there is at least one decision node that succeeds x. Two cases are possible: (1) there is no information set that is maximal relative to x, (2) there is at least one maximal information set relative to x. In case (1) fix an arbitrary arc (x, y) and let B be the corresponding action [that is, $\xi((x, y)) = B$]. Let v be the information set to which y belongs. Let (x', y') be the arc corresponding to action B in G' and let v' be the information set in G' that corresponds to v. We want to show that $y' \in v'$. Suppose not, that is, suppose $y' \in w' \neq v'$. Then in G' all the plays that have $A_1, \ldots, A_m$ as first actions cross information set w'. Let w be the information set in G that corresponds to w'. By our supposition $w \neq v$. Thus $x \notin w$ and x precedes w. By lemma 3.1, also in G all plays that have $A_1, \ldots, A_m$ as first actions must cross w. But those plays must go through node x, which implies that w is a maximal information set relative to x, a contradiction. Thus if there is no maximal information set relative to x, the paths of length $(m+1)$ that go through x and x' coincide in G and G'. Suppose now that we are in case (2) and there are r information sets that are maximal relative to x. Then by lemma 3.1 there are r information sets in G' that are maximal relative to x'. Then we can apply the algorithm of lemma 6.2: fix an ordering of the corresponding situations and apply ICSM to transform G and G' into new games where the paths of length $(m+1)$ through x and x' coincide.

STEP 3. For every node x in the new game obtained from G at the end of step 2 and corresponding node x' in the new game obtained from G' (at the end of step 2), such that the paths through x and x' are identical in the two games, apply again the procedure of step 2. Repeat until the same game is obtained from both games. □

Figure 6 illustrates an equivalence class. The arrows show the result of applying (once) the transformation of interchange of contiguous simultaneous moves to any given game in the class.

7 Conclusion

We introduced a new game-form, called the set-theoretic form, and showed that a set-theoretic form can be associated with every extensive form. Since the map from extensive forms to set-theoretic forms is not one-to-one we used this fact to define a notion of equivalence of extensive games. We then described a transformation, called the interchange of contiguous simultaneous moves, and showed that it is possible to move from one game to any other game in the same equivalence class by using this transformation a finite number of times and without ever leaving the equivalence class. This transformation is a generalization of Thompson's "interchange of decision nodes". Thus given an extensive game G there is a different extensive game G' that is equivalent to G if any only if there are moves in G that are simultaneous and the difference between G and G' lies exactly in the fact that (some of) these moves are taken in a different temporal order in the two games.

Consider now the equivalence classes of extensive games generated by the relations of set-theoretic equivalence. These equivalence classes have the same richness of the extensive form, excluding only the specification of the temporal order in which simultaneous moves are made. An interesting open question is the following: what extensive-form solution concepts are invariant with respect to the equivalence classes generated by the notion of set-theoretic form? For example, does the notion of sequential equilibrium have this invariance property?

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