Royal Netherlands Academy of Arts and Sciences (KNAW)
Master Class

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Epistemic Foundations of Game Theory

Lecture 1

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QUESTION:

What strategies can be chosen by ***rational*** players who ***know*** the structure of the game and the preferences of their opponents and who ***recognize*** each other's rationality and knowledge?

Keywords: knowledge, rationality, recognition of each other's knowledge and rationality

Modular approach

Module 1: representation of belief and knowledge of an individual (Hintikka, 1962; Kripke, 1963).

Module 2: extension to many individuals.
Common belief and common knowledge
(“recognition of each other’s belief / knowledge”)

Module 3: definition of rationality in games
(relationship between choice and beliefs)

QUESTION: **what are the implications of rationality and common belief of rationality in games?**

Module 1

representation of beliefs and knowledge of an individual

Finite set of states Ω and a binary relation $\mathcal{B}$ on Ω.

$\alpha \mathcal{B} \beta$ means "at state α the individual considers state β possible"

Notation: $\mathcal{B}(\omega) = \{\omega' \in \Omega : \omega \mathcal{B} \omega'\}$ set of states considered possible at ω

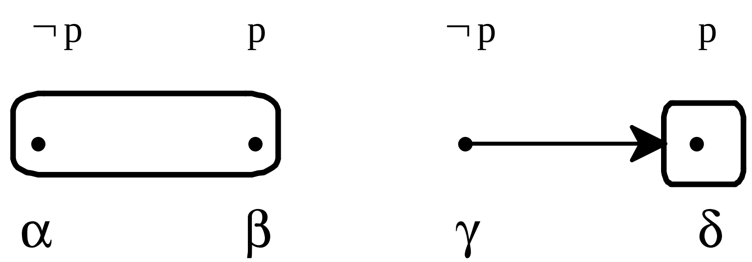
PROPERTIES $\forall \omega, \omega' \in \Omega,$

1. $\mathcal{B}(\omega) \neq \varnothing$ seriality
2. if $\omega' \in \mathcal{B}(\omega)$ then $\mathcal{B}(\omega') \subseteq \mathcal{B}(\omega)$ transitivity
3. if $\omega' \in \mathcal{B}(\omega)$ then $\mathcal{B}(\omega) \subseteq \mathcal{B}(\omega')$ euclideannes

Belief operator on events: $B : 2^{\Omega} \to 2^{\Omega}$

For $E \subseteq \Omega$, $\omega \in BE$ *if and only if* $\mathcal{B}(\omega) \subseteq E$

EXAMPLE:



$$\mathcal{B}(\alpha) = \mathcal{B}(\beta) = \{\alpha, \beta\}$$
$$\mathcal{B}(\gamma) = \mathcal{B}(\delta) = \{\delta\}$$

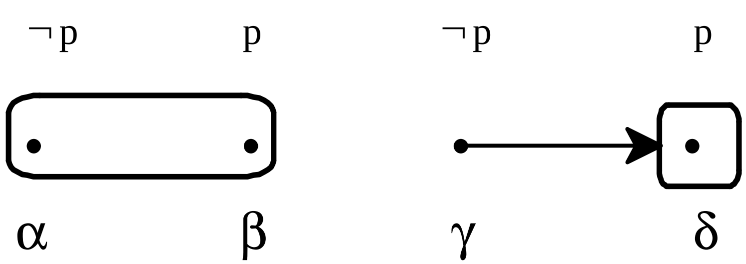
Let $E = \{\beta, \delta\}$: the event that represents the proposition p

Then $BE = \{\gamma, \delta\}$

Properties of the belief operator: $\forall E \subseteq \Omega$

1. $BE \subseteq \neg B \neg E$ (consistency: follows from seriality of $\mathcal{B}$)
2. $BE \subseteq BBE$ (positive introspection: follows from transitivity of $\mathcal{B}$)
3. $\neg BE \subseteq B \neg BE$ (negative introspection: follows from euclideanness of $\mathcal{B}$)

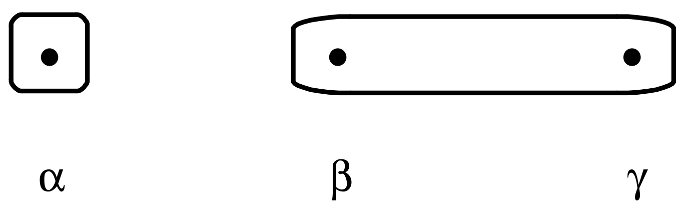
Mistaken beliefs are possible: at γ *p* is false but the individual believes *p*



If $E = \{\beta, \delta\}$, then
$\gamma \notin E$ but $\gamma \in BE = \{\gamma, \delta\}$

KNOWLEDGE

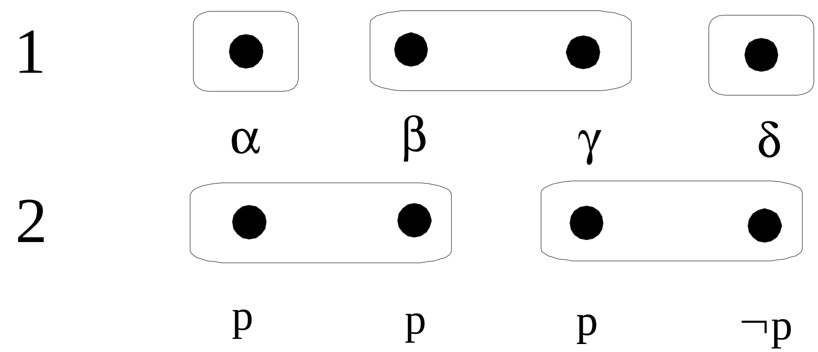
If - in addition to the previous properties - the "doxastic accessibility" relation $\mathcal{B}$ is ***reflexive*** ($\forall \omega \in \Omega,\ \omega \in \mathcal{B}(\omega)$) then it is an ***equivalence relation*** - giving rise to a ***partition*** of the set of states - and the associated belief operator satisfies the additional property that $\forall E \subseteq \Omega,\ \ BE \subseteq E$ (beliefs are correct). In this case we speak of ***knowledge*** and the associated operator is denoted by K rather than B



Module 2

interactive belief and common belief

Set of individuals N and a binary relation $\mathcal{B}_i$ for every $i \in N$



Let $E = \{\alpha, \beta, \gamma\}$: the event that represents the proposition p

Then $K_1 E = \{\alpha, \beta, \gamma\}$, $K_2 E = \{\alpha, \beta\}$

$K_1 K_2 E = \{\alpha\}$, $K_2 K_1 K_2 E = \varnothing$

An event E is ***commonly believed*** if (1) everybody believes it, (2) everybody believes that everybody believes it, (3) everybody believes that everybody believes that everybody believes it, etc.

Define the "everybody believes" operator B^e as follows:

$$B^e E = B_1 E \cap B_2 E \cap \ldots \cap B_n E$$

The common belief operator B_* is defined as follows:

$$B_* E = B^e E \cap B^e B^e E \cap B^e B^e B^e E \cap \ldots$$

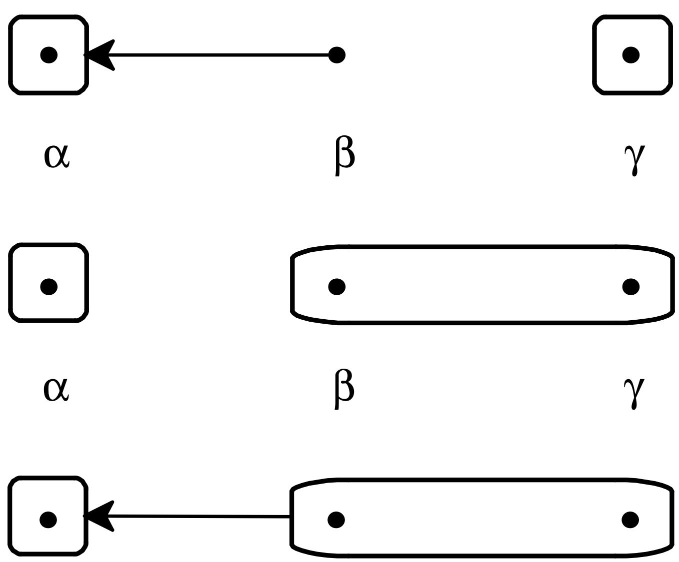
Let $\mathcal{B}_*$ be the *transitive closure* of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_n$

Thus $\omega' \in \mathcal{B}_*(\omega)$ if and only if there exists a sequence $\langle \omega_1, \ldots, \omega_m \rangle$ in Ω such that

(1) $\omega_1 = \omega$

(2) $\omega_n = \omega'$

(3) for every $j = 1, \ldots, m$ there exists an individual $i \in N$ such that $\omega_{j+1} \in \mathcal{B}_i(\omega_j)$

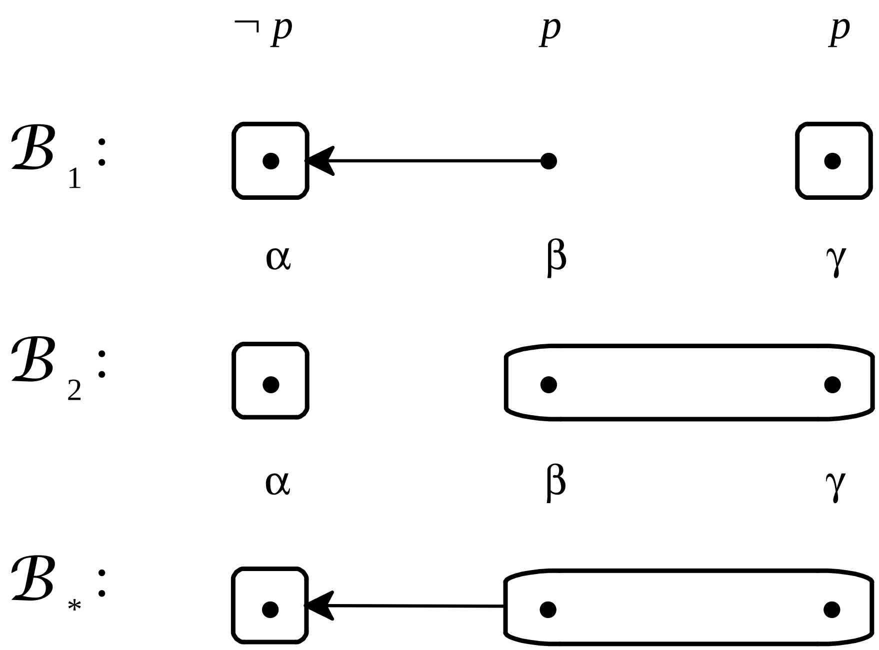


$\mathcal{B}_1$: $\mathcal{B}_1(\alpha) = \mathcal{B}_1(\beta) = \{\alpha\}, \ \mathcal{B}_1(\gamma) = \{\gamma\}$

$\mathcal{B}_2$: $\mathcal{B}_2(\alpha) = \{\alpha\}, \ \mathcal{B}_2(\beta) = \mathcal{B}_2(\gamma) = \{\beta, \gamma\}$

$\mathcal{B}_*$: $\mathcal{B}_*(\alpha) = \{\alpha\}, \ \mathcal{B}_*(\beta) = \mathcal{B}_*(\gamma) = \{\alpha, \beta, \gamma\}$

PROPOSITION. $\omega \in B_* E$ if and only if $\mathcal{B}_*(\omega) \subseteq E$.



Let $E = \{\beta, \gamma\}$: the event that represents the proposition p

Then $B_1 E = \{\gamma\}$, $B_2 E = \{\beta, \gamma\}$, $B_* E = \varnothing$

In fact, while $\gamma \in B_1 B_2 E = \{\gamma\}$, $\gamma \notin B_2 B_1 E = \varnothing$

Module 3
Models of games and Rationality

Definition. A finite strategic-form game ***with ordinal payoffs*** is a quintuple

$$\left\langle N, \{S_i\}_{i\in N}, O, \{\succsim_i\}_{i\in N}, z \right\rangle$$

$N = \{1,\dots,n\}$ is a set of *players*

S_i is a finite set of *strategies* or choices of player $i \in N$

O is a set of *outcomes*

$\succsim_i$ is player i's ordering of O ($o \succsim_i o'$ means that, for player i, outcome o is at least as good as outcome o')

$z : S \to O$ (where $S = S_1 \times \dots \times S_n$) associates an outcome with every strategy profile $s \in S$

Definition. Given a strategic-form game with ordinal payoffs

$$\left\langle N, \{S_i\}_{i\in N}, O, \{\succsim_i\}_{i\in N}, z \right\rangle$$

a reduced form of it is a triple

$$\left\langle N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N} \right\rangle$$

where $u_i : S \to \mathbb{R}$ is such that $u_i(s) \geq u_i(s')$ if and only if $z(s) \succsim_i z(s')$

player i's *utility function*

Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	1 , 2	4 , 1
D	0 , 2	0 , 3	1 , 3

SAME AS

Player 2

Player 1	e	f	g
A	9 , 6	6 , 4	0 , 4
B	4 , 9	3 , 3	2 , 0
C	2 , 5	2 , 5	8 , 2
D	1 , 0	0 , 8	1 , 8

Definition. An *epistemic model* of a strategic-form game is an interactive belief structure together with *n* functions

$$\sigma_i : \Omega \to S_i \quad (i \in N)$$

Interpretation: $\sigma_i(\omega)$ is player *i*'s chosen strategy at state ω

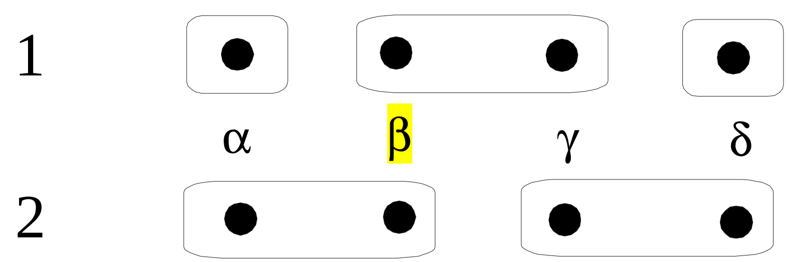
Restriction: if $\omega' \in \mathcal{B}_i(\omega)$ then $\sigma_i(\omega') = \sigma_i(\omega)$

(no player has mistaken beliefs about her own strategy)

EXAMPLE

Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	1 , 2	4 , 1
D	0 , 2	0 , 3	1 , 3



1's strategy:	A	C	C	D
2's strategy:	f	f	g	g

At every state each player knows his own strategy

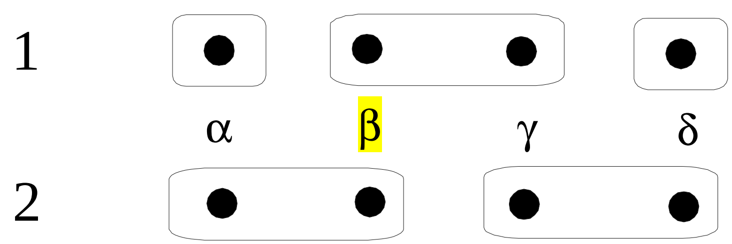
At state β player 1 plays C (he knows this) not knowing whether player 2 is playing f or g and player 2 plays f (she knows this) not knowing whether player 1 is playing A or C

RATIONALITY

Non-probabilistic (no expected utility) and very weak notion of rationality

Definition. Player i is ***IRRATIONAL*** at state ω if there is a strategy s_i (of player i) which she believes to be better than $\sigma_i(\omega)$ (that is, if she believes that she can do better with another strategy)

Player i is ***RATIONAL*** at state ω if and only if she is not irrational



1's strategy:	A	C	C	D
2's strategy:	f	f	g	g

Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	1 , 2	4 , 1
D	0 , 2	0 , 3	1 , 3

Player 1 is rational at state β

Let s_i and t_i be two strategies of player i: $s_i, t_i \in S_i$

$s_i \succ_i t_i$ is interpreted as "strategy s_i is better for player i than strategy t_i"

$s_i \succ_i t_i$ is true at state ω if $u_i(s_i, \sigma_{-i}(\omega)) > u_i(t_i, \sigma_{-i}(\omega))$

that is, s_i is better than t_i against $\sigma_{-i}(\omega)$

profile of strategies chosen by the players other than i

	α	β	γ
	●	●	●
1's strategy:	A	C	C
2's strategy:	E	F	G

Player 2

Player 1	E	F	G
A	3 , 2	1 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	0 , 2	4 , 1

$A \succ_1 B$ $\quad B \succ_1 A$ $\quad C \succ_1 B$

$A \succ_1 C$ $\quad B \succ_1 C$ $\quad C \succ_1 A$

$B \succ_1 C$ $\quad A \succ_1 C$ $\quad B \succ_1 A$

$E \succ_2 F$ $\quad F \succ_2 G$ $\quad F \succ_2 G$ $\quad$ etc.

Let $\|s_i \succ_i t_i\| = \{\omega \in \Omega : u_i(s_i, \sigma_{-i}(\omega)) > u_i(t_i, \sigma_{-i}(\omega))\}$ event that s_i is better than t_i

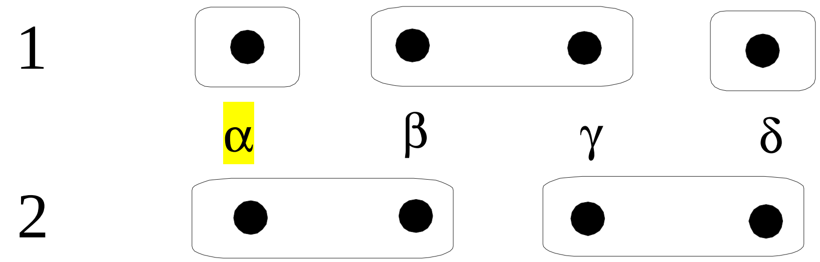
If $s_i \in S_i$, let $\|s_i\| = \{\omega \in \Omega : \sigma_i(\omega) = s\}$ event that player i chooses s_i

Let $\boldsymbol{R_i}$ be the event representing the proposition "player i is rational"

$$\|s_i\| \cap B_i \|t_i \succ_i s_i\| \subseteq \neg \boldsymbol{R_i}$$

$$\neg \boldsymbol{R_i} = \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} \left(\|s_i\| \cap B_i \|t_i \succ_i s_i\| \right)$$

$$\boldsymbol{R} = \boldsymbol{R_1} \cap \ldots \cap \boldsymbol{R_n}$$ all players are rational



	α	β	γ	δ
1's strategy:	A	C	C	D
2's strategy:	f	f	g	g
	R_1	R_1	R_1	$\neg R_1$
	R_2	R_2	R_2	R_2
	K_1R_2	K_1R_2	K_1R_2	K_1R_2
	K_2R_1	K_2R_1	$\neg K_2R_1$	$\neg K_2R_1$
	$K_1K_2R_1$	$\neg K_1K_2R_1$	$\neg K_1K_2R_1$	$\neg K_1K_2R_1$
	$\neg K_2K_1K_2R_1$			

Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	1 , 2	4 , 1
D	0 , 2	0 , 3	1 , 3

$$\boldsymbol{R_1} = \{\alpha, \beta, \gamma\}, \quad \boldsymbol{R_2} = \{\alpha, \beta, \gamma, \delta\}$$

$$K_1\boldsymbol{R_2} = \{\alpha, \beta, \gamma, \delta\}, \ K_2\boldsymbol{R_1} = \{\alpha, \beta\}$$

$$K_1K_2\boldsymbol{R_1} = \{\alpha\}, \quad K_2K_1K_2\boldsymbol{R_1} = \varnothing$$

At state α there is mutual knowledge of rationality but not common knowledge of rationality

Let $S_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$ set of strategy profiles of all players except *i*

Definition. Let $s_i, t_i \in S_i$. We say that t_i is *strictly dominated* by s_i if $u_i(t_i, s_{-i}) < u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$

ITERATED DELETION OF STRICTLY DOMINATED STRATEGIES

Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	1 , 2	4 , 1
~~D~~	~~0 , 2~~	~~0 , 3~~	~~1 , 3~~

(by C)

Player 2

	e	f	~~g~~
A	3 , 2	3 , 1	~~0 , 1~~
B	2 , 3	2 , 2	~~3 , 1~~
C	1 , 2	1 , 2	~~4 , 1~~

(by e)

	e	f
A	3 , 2	3 , 1
B	2 , 3	2 , 2
~~C~~	~~1 , 2~~	~~1 , 2~~

(by B)

	e	~~f~~
A	3 , 2	~~3 , 1~~
B	2 , 3	~~2 , 2~~

(by e)

	e
A	3 , 2
~~B~~	~~2 , 3~~

(by A)

Let G be a strategic-form game with ordinal payoffs and G^∞ be the game obtained after applying the procedure of Iterated Deletion of Strictly Dominated Strategies.

Let S^∞ denote the strategy profiles of game G^∞

Given a model of G, let $\mathbf{S}^\infty$ denote the event $\{\omega \in \Omega : \sigma(\omega) \in S^\infty\}$

G

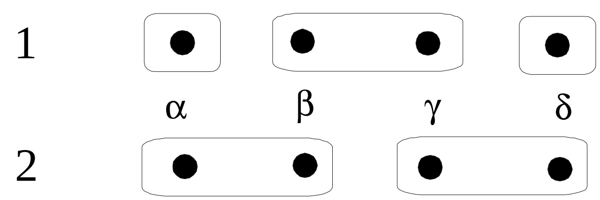
Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	1 , 2	4 , 1
D	0 , 2	0 , 3	1 , 3

G^∞

	e
A	3 , 2

$S^\infty = \{(A, e)\}$

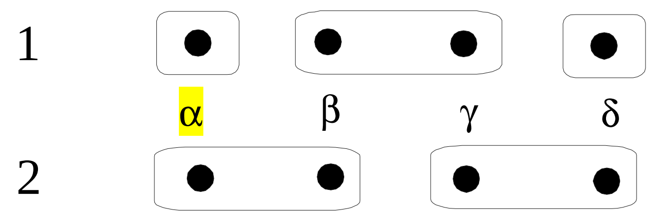


	α	β	γ	δ
1's strategy:	A	C	C	A
2's strategy:	f	f	e	e

$\mathbf{S}^\infty = \{\delta\}$

PROPOSITION 1. $B_* R \subseteq S^\infty$

If at a state it is commonly believed that all players are rational, then the strategy profile chosen at that state belongs to the game obtained after applying the iterated deletion of strictly dominated strategies.



	α	β	γ	δ
1's strategy:	A	C	C	D
2's strategy:	f	f	g	g
	R_1	R_1	R_1	$\neg R_1$
	R_2	R_2	R_2	R_2
	K_1R_2	K_1R_2	K_1R_2	K_1R_2
	K_2R_1	K_2R_1	$\neg K_2R_1$	$\neg K_2R_1$
	$K_1K_2R_1$	$\neg K_1K_2R_1$	$\neg K_1K_2R_1$	$\neg K_1K_2R_1$
	$\neg K_2K_1K_2R_1$			

Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	1 , 2	4 , 1
D	0 , 2	0 , 3	1 , 3

At state α there cannot be common knowledge of rationality since $\sigma(\alpha) \neq (A, e)$

Every normal operator B satisfies the property that if $E \subseteq F$ then $BE \subseteq BF$.

B_* is a normal operator. Thus from $B_*\boldsymbol{R} \subseteq \boldsymbol{S}^\infty$ it follows that $B_*B_*\boldsymbol{R} \subseteq B_*\boldsymbol{S}^\infty$.

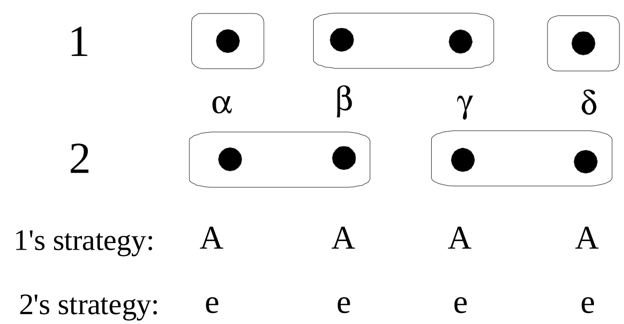
By transitivity of $\mathcal{B}_*$ we have that

$B_*E \subseteq B_*B_*E$ for every event E.

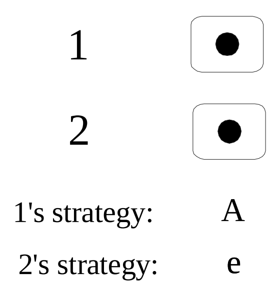
Thus $B_*\boldsymbol{R} \subseteq B_*B_*\boldsymbol{R}$.

It follows that $B_*\boldsymbol{R} \subseteq B_*\boldsymbol{S}^\infty$

		Player 2		
		e	f	g
Player 1	A	3 , 2	3 , 1	0 , 1
	B	2 , 3	2 , 2	3 , 1
	C	1 , 2	1 , 2	4 , 1
	D	0 , 2	0 , 3	1 , 3



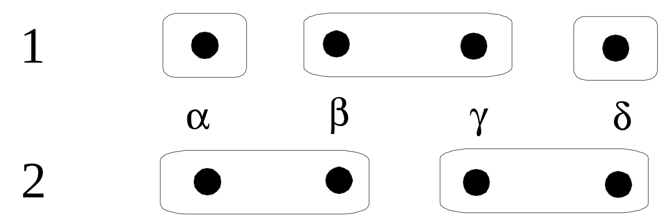
Same as:



REMARK. In general it is not true that $\boldsymbol{S}^{\infty} \subseteq B_{*}\boldsymbol{R}$

Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	2 , 2	3 , 1
C	1 , 2	1 , 2	4 , 1
D	0 , 2	0 , 3	1 , 3



1's strategy:	A	C	C	A
2's strategy:	f	f	e	e

$$\mathbf{S}^{\infty} = \{\delta\}$$

$$K_{*}\boldsymbol{R} = \varnothing$$

$$\boldsymbol{R}_1 = \{\alpha, \delta\}, \quad \boldsymbol{R}_2 = \{\alpha, \beta, \gamma, \delta\}$$

$$K_2 \boldsymbol{R}_1 = \varnothing$$

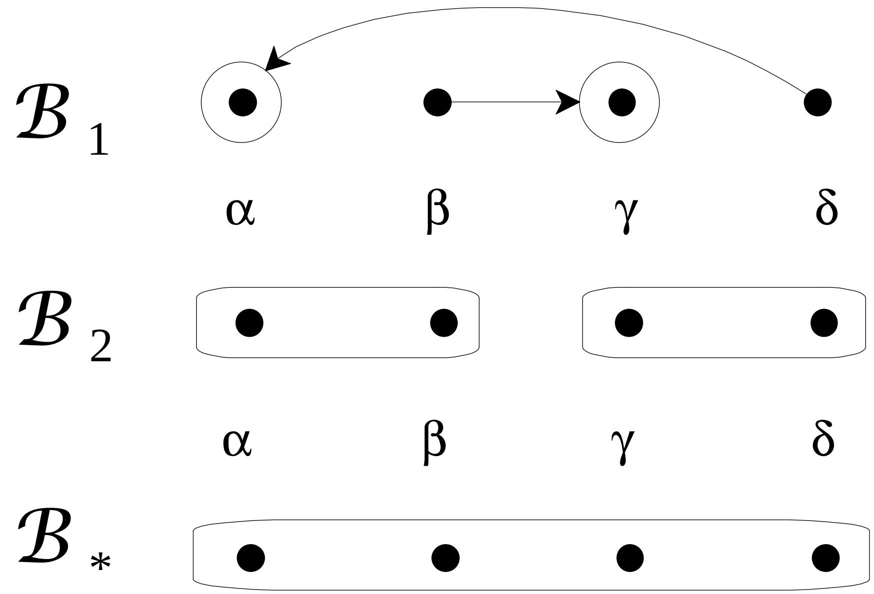
PROPOSITION 2. Fix a strategic-form game with ordinal payoffs G and let $s \in S^{\infty}$. Then there exists an epistemic model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in B_* \boldsymbol{R}$.

EXAMPLE

Player 2

Player 1		e	f
	A	3 , 2	3 , 3
	B	2 , 3	4 , 2

In this game every strategy profile survives iterative deletion



σ_1	A	B	B	A
σ_2	e	e	f	f

In this model $\boldsymbol{R} = B_* \boldsymbol{R} = \Omega$ and every strategy profile occurs at some state

REMARK. Given the above notion of rationality, *there is no difference between common belief of rationality and common knowledge of rationality*. The previous two propositions can be restated in terms of knowledge and common knowledge.

PROPOSITION 1′. $K_* \boldsymbol{R} \subseteq \boldsymbol{S}^\infty$

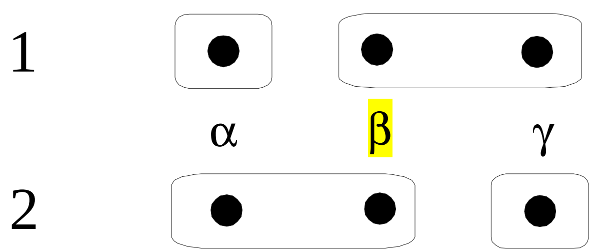
PROPOSITION 2′. Fix a strategic-form game with ordinal payoffs G and let $s \in S^\infty$. Then there exists an epistemic model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in K_* \boldsymbol{R}$.

STRONGER NOTION OF RATIONALITY

Still non-probabilistic (no expected utility)

Definition. Player i is ***IRRATIONAL*** at state ω if there is a strategy s_i which she believes to be at least as good as $\sigma_i(\omega)$ and she considers it possible that s_i is better than $\sigma_i(\omega)$

Player i is ***RATIONAL*** at state ω if and only if she is not irrational



1's strategy:	A	C	C
2's strategy:	f	f	g

Player 2

Player 1	e	f	g
A	3 , 2	3 , 1	0 , 1
B	2 , 3	1 , 2	3 , 1
C	4 , 2	1 , 2	2 , 1

Player 1 is **ir**rational at state β: *B is at least as good as C at both* β *and* γ *and it is better than C at* γ

$$\boldsymbol{R_1} = \{\alpha\}, \ \ \boldsymbol{R_2} = \varnothing$$

Player i is ***IRRATIONAL*** at state ω if there is a strategy s_i which she believes to be at least as good as $\sigma_i(\omega)$ and she considers it possible that s_i is better than $\sigma_i(\omega)$

$$\|s_i\| \cap B_i \|t_i \succsim_i s_i\| \cap \neg B_i \neg \|t_i \succ_i s_i\| \subseteq \neg \boldsymbol{R_i}$$

$$\neg \boldsymbol{R_i} = \bigcup_{s_i \in S_i} \bigcup_{t_i \in S_i} \left(\|s_i\| \cap B_i \|t_i \succsim_i s_i\| \cap \neg B_i \neg \|t_i \succ_i s_i\| \right)$$

$$\boldsymbol{R} = \boldsymbol{R_1} \cap \ldots \cap \boldsymbol{R_n}$$ all players are rational

Definition.

Given a game $G = \left\langle N, \{S_i\}_{i\in N}, O, \{\succsim\}_{i\in N}, z \right\rangle$, a subset of strategy profiles $X \subseteq S$ and a strategy profile $x \in X$, we say that x is **inferior relative to** X if there exist a player i and a strategy $s_i \in S_i$ of player i (thus s_i need not belong to the projection of X onto S_i) such that:

1. $z(s_i, x_{-i}) \succ_i z(x_i, x_{-i})$ and
2. for all $s_{-i} \in S_{-i}$, if $(x_i, s_{-i}) \in X$ then $z(s_i, s_{-i}) \succsim_i z(x_i, s_{-i})$.

Iterated Deletion of Inferior Profiles: for $m \in \mathbb{N}$ define $T^m \subseteq S$ recursively as follows: $T^0 = S$ and, for $m \geq 1$, $T^m = T^{m-1} \setminus I^{m-1}$, where $I^{m-1} \subseteq T^{m-1}$ is the set of strategy profiles that are inferior relative to T^{m-1}. Let $T^\infty = \bigcap_{m\in\mathbb{N}} T^m$.

Player 2

Player 1		d	e	f
	A	2 , 1	0 , 1	2 , 1
	B	1 , 0	1 , 0	1 , 1
	C	1 , 4	1 , 3	0 , 3

T^0

Player 2

Player 1		d	e	f
	A	2 , 1	0 , 1	2 , 1
	B	1 , 0		1 , 1
	C	1 , 4	1 , 3	

T^1

Player 2

Player 1		d	e	f
	A	2 , 1	0 , 1	2 , 1
	B			
	C	1 , 4		

T^2

Player 2

Player 1		d	e	f
	A	2 , 1	0 , 1	2 , 1
	B			
	C			

$T^\infty = T^3$

$T^0 = S = \{(A,d),(A,e),(A,f),(B,d),(B,e),(B,f),(C,d),(C,e),(C,f)\}$, $I^0 = \{(B,e),(C,f)\}$ (the elimination of (B,e) is done through player 2 and strategy f, while the elimination of (C,f) is done through player 1 and strategy B);

$T^1 = \{(A,d),(A,e),(A,f),(B,d),(B,f),(C,d),(C,e)\}$, $I^1 = \{(B,d),(B,f),(C,e)\}$ (the elimination of (B,d) and (B,f) is done through player 1 and strategy A, while the elimination of (C,e) is done through player 2 and strategy d);

$T^2 = \{(A,d),(A,e),(A,f),(C,d)\}$, $I^2 = \{(C,d)\}$ (the elimination of (C,d) is done through player 1 and strategy A);

$T^3 = \{(A,d),(A,e),(A,f)\}$, $I^3 = \varnothing$; thus $T^\infty = T^3$.

PROPOSITION 3. $K_* \boldsymbol{R} \subseteq \boldsymbol{T}^{\infty}$

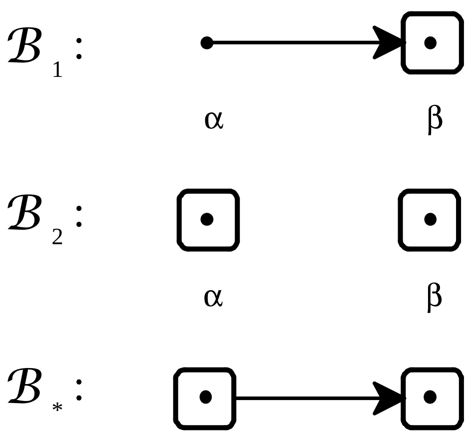
If at a state it is commonly **known** that all players are rational, then the strategy profile chosen at that state belongs to the game obtained after applying the iterated deletion of Inferior strategy profiles.

PROPOSITION 4. Fix a strategic-form game with ordinal payoffs G and let $s \in T^{\infty}$. Then there exists an epistemic model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in K_* \boldsymbol{R}$.

NOT TRUE if we replace common knowledge with common belief

		Player 2 c	d
Player 1	A	1 , 1	1 , 0
	B	1 , 1	0 , 1

$$\boldsymbol{R_1} = \{\alpha, \beta\}, \quad \boldsymbol{R_2} = \{\alpha, \beta\}$$



σ_1 :	B	B
σ_2 :	d	c

There is common belief of rationality at every state and yet at state α the strategy profile played is (B,d) which is inferior

$$\boldsymbol{T^{\infty} = \{(A,c),(B,c)\}}$$

$$\boldsymbol{S^{\infty} = \{(A,c),(A,d),(B,c),(B,d)\}}$$

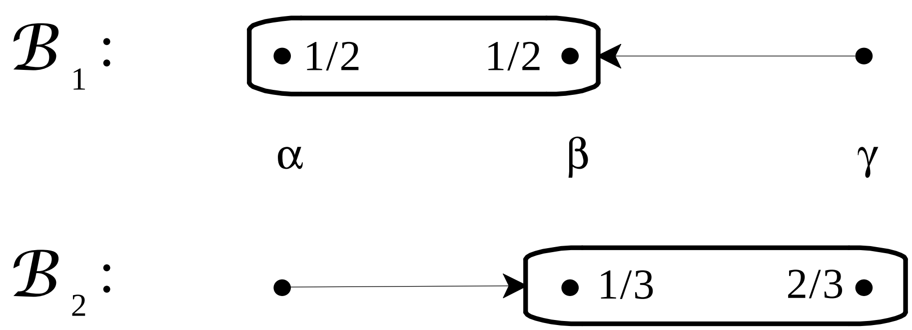
PROBABILISTIC BELIEFS

Definition. A *Bayesian frame* is an interactive belief frame together with a collection $\{p_{i,\omega}\}_{i\in N,\ \omega\in\Omega}$ of probability distributions on Ω such that

(1) if $\omega' \in \mathcal{B}_i(\omega)$ then $p_{i,\omega'} = p_{i,\omega}$

(2) $p_{i,\omega}(\omega') > 0$ if and only if $\omega' \in \mathcal{B}_i(\omega)$

(the support of $p_{i,\omega}$ coincides with $\mathcal{B}_i(\omega)$)



Definition. A strategic-form game ***with von Neumann-Morgenstern payoffs*** is a quintuple

$$\left\langle N, \{S_i\}_{i\in N}, O, \{U_i\}_{i\in N}, z \right\rangle$$

where

$N = \{1,...,n\}$ is a set of *players*

S_i is the set of *strategies* of player $i \in N$

O is a set of *outcomes*

$U_i : O \to \mathbb{R}$ is player *i*'s von Neumann-Morgenstern utility function

$z : S \to O$ (where $S = S_1 \times ... \times S_n$) associates an outcome with every strategy profile $s \in S$

Its reduced form is a triple $\left\langle N, \{S_i\}_{i\in N}, \{\pi_i\}_{i\in N} \right\rangle$ where $\pi_i(s) = U_i(z(s))$.

An *epistemic model* of a strategic-form game is a Bayesian frame together with *n* functions

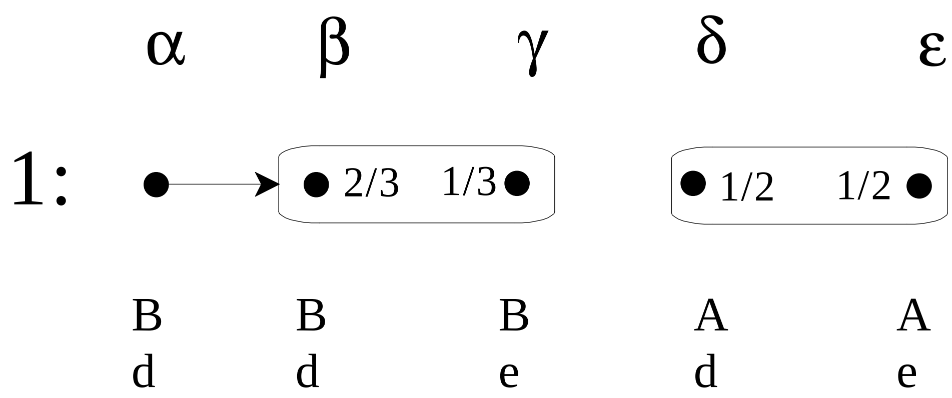
$$\sigma_i : \Omega \to S_i \quad (i \in N)$$

such that if $\omega' \in \mathcal{B}_i(\omega)$ then $\sigma_i(\omega') = \sigma_i(\omega)$

Stronger definition of Rationality than the previous ones

Player *i* is **RATIONAL** at state α if her choice at α maximizes her expected payoff, given her beliefs at α: for all $t_i \in S_i$

$$\sum_{\omega \in \mathcal{B}_i(\alpha)} \pi_i\left(\sigma_i(\alpha), \sigma_{-i}(\omega)\right) p_{i,\alpha}(\omega) \geq \sum_{\omega \in \mathcal{B}_i(\alpha)} \pi_i\left(t_i, \sigma_{-i}(\omega)\right) p_{i,\alpha}(\omega)$$



		Player 2: d	Player 2: e
Player 1	A	3 , 0	0 , 1
	B	1 , 0	2 , 2
	C	0 , 3	3 , 2

$$\boldsymbol{R_1} = \{\delta, \varepsilon\}$$

Player 1 is not rational at α because her expected payoff is $\frac{2}{3}1+\frac{1}{3}2=\frac{4}{3}$

while if she had chosen strategy A her payoff would have been $\frac{2}{3}3+\frac{1}{3}0=2$

On the other hand, Player 1 *is* rational at δ because her expected payoff is $\frac{1}{2}3+\frac{1}{2}0=\frac{3}{2}$

and if she had chosen strategy B her payoff would have been $\frac{1}{2}1+\frac{1}{2}2=\frac{3}{2}$

and if she had chosen strategy C her payoff would have been $\frac{1}{2}0+\frac{1}{2}3=\frac{3}{2}$

What are the implications of Common Belief of this stronger notion of rationality?

Definition. A mixed strategy of player i is a probability distribution over S_i
The set of mixed strategies of player i is denoted by $\Delta(S_i)$

Let $t_i \in S_i$ and $\nu_i \in \Delta(S_i)$. We say that t_i is *strictly dominated* by ν_i if,

for every $s_{-i} \in S_{-i}$, $\pi_i(t_i, s_{-i}) < \sum_{s_i \in S_i} \nu_i(s_i)\pi_i(s_i, s_{-i})$

Player 2

Player 1		d	e
	A	3 , 0	0 , 1
	B	0 , 0	2 , 2
	C	0 , 3	3 , 2

In this game strategy B of player 1 is strictly dominated by the mixed strategy $\begin{pmatrix} A & C \\ \frac{1}{6} & \frac{5}{6} \end{pmatrix}$

ITERATIVE DELETION OF PURE STRATEGIES THAT ARE STRICTLY DOMINATED BY (POSSIBLY MIXED) STRATEGIES

Player 2

Player 1	e	f	g
A	3 , 0	1 , 0	0 , 1
B	1 , 1	0 , 2	1 , 1
C	0 , 0	4 , 1	2 , 2
D	0 , 3	1 , 0	3 , 2

(a) The game G

B is strictly dominated by (1/2 A, 1/2 D)

Player 2

Player 1	e	f	g
A	3 , 0	1 , 0	0 , 1
C	0 , 0	4 , 1	2 , 2
D	0 , 3	1 , 0	3 , 2

(b) The game G^1

Now f is strictly dominated by g

Player 2

Player 1	e	g
A	3 , 0	0 , 1
C	0 , 0	2 , 2
D	0 , 3	3 , 2

(c) The game G^2

Now C is strictly dominated by (1/6 A, 5/6 D)

Player 2

Player 1	e	g
A	3 , 0	0 , 1
D	0 , 3	3 , 2

(d) The game $G^3 = G^\infty$

No strategy is strictly dominated

Let G be a strategic-form game with von Neumann-Morgenstern payoffs and G^{∞} be the game obtained after applying the procedure of Iterated Deletion of Pure Strategies that are Strictly Dominated by Possibly Mixed Strategies.

Let S_m^{∞} denote the pure-strategy profiles of game G^{∞}

Given a model of G, let $\boldsymbol{S}_{\boldsymbol{m}}^{\infty}$ be the event $\left\{\omega \in \Omega : \sigma(\omega) \in \boldsymbol{S}_{\boldsymbol{m}}^{\infty}\right\}$

PROPOSITION 5. $B_{*}\boldsymbol{R} \subseteq \boldsymbol{S}_{\boldsymbol{m}}^{\infty}$

PROPOSITION 6. Fix a strategic-form game with von Neumann-Morgenstern payoffs G and let $s \in S_m^{\infty}$. Then there exists a Bayesian model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in B_{*}\boldsymbol{R}$.

Given this stronger notion of rationality, *there **is** a difference between common belief of rationality and common knowledge of rationality*. The implications of common knowledge of rationality are stronger.

With knowledge, a player's beliefs are always correct and are believed to be correct by every other player. Thus there is *correctness and common belief of correctness* of everybody's beliefs.

Definition. Given a strategic-form game with von Neumann-Morgenstern payoffs G, a pure-strategy profile $x \in X \subseteq S$ is *inferior relative to* X if there exists a player i and a (possibly mixed) strategy ν_i of player i (whose support can be any subset of S_i, not necessarily the projection of X onto S_i) such that:

(1) $\pi_{\mathrm{i}}(x_i, x_{-i}) < \sum_{s_i \in S_i} \pi_i(s_i, x_{-i}) \nu_i(s_i)$ (ν_i yields a higher expected payoff than x_i against x_{-i})

(2) for all $s_{-i} \in S_{-i}$ such that $(x_i, s_{-i}) \in X$, $\pi_{\mathrm{i}}(x_i, s_{-i}) \leq \sum_{s_i \in S_i} \pi_i(s_i, s_{-i}) \nu_i(s_i)$

		Player 2		
		D	E	F
Player 1	A	2 , 0	2 , 2	0 , 2
	B	2 , 2	1 , 2	5 , 1
	C	2 , 0	1 , 0	1 , 5

Here (C,F) is inferior relative to S (for player 1, B weakly dominates C and is strictly better than C against F)

and (A,D) is inferior relative to S (for player 2, E weakly dominates D and is strictly better than D against A)

ITERATED DELETION OF INFERIOR PURE STRATEGY PROFILES

Player 2

		D	E	F
Player	A	2 , 0	2 , 2	0 , 2
1	B	2 , 2	1 , 2	5 , 1
	C	2 , 0	1 , 0	1 , 5

(a)

$S_s^0 = S,\ D_s^0 = \{(A, D), (C, F)\}$

Player 2

		D	E	F
Player	A		2 , 2	0 , 2
1	B	2 , 2	1 , 2	5 , 1
	C	2 , 0	1 , 0	

(b)

$S_s^1 = \{(A, E), (A, F), (B,D), (B, E), (B, F), (C, D), (C, E)\}$

$D_s^1 = \{(C, E), (B, F)\}$

Player 2

		D	E	F
Player	A		2 , 2	0 , 2
1	B	2 , 2	1 , 2	
	C	2 , 0		

(c)

$S_s^2 = \{(A, E), (A, F), (B,D), (B, E), (C, D)\},$

$D_s^2 = \{(B, E)\}.$

Player 2

		D	E	F
Player	A		2 , 2	0 , 2
1	B	2 , 2		
	C	2 , 0		

(d)

$S_s^3 = S_s^\infty = \{(A, E), (A, F), (B,D), (C, D)\}, \quad D_s^3 = \varnothing.$

Let G be a strategic-form game with von Neumann-Morgenstern payoffs and G^{∞} be the game obtained after applying the procedure of Iterated Deletion of Inferior Pure-Strategy Profiles.

Let S_s^{∞} denote the pure-strategy profiles of game G^{∞}

Given a model of G, let $\boldsymbol{S}_s^{\infty}$ be the event $\{\omega \in \Omega : \sigma(\omega) \in \boldsymbol{S}_s^{\infty}\}$

PROPOSITION 7. $K_* \boldsymbol{R} \subseteq \boldsymbol{S}_s^{\infty}$

PROPOSITION 8. Fix a strategic-form game with von Neumann-Morgenstern payoffs G and let $s \in S_s^{\infty}$. Then there exists a Bayesian model of G and a state ω such that $\sigma(\omega) = s$ and $\omega \in K_* \boldsymbol{R}$.

			Player 2	
		D	E	F
Player 1	A	2 , 0	2 , 2	0 , 2
	B	2 , 2	1 , 2	5 , 1
	C	2 , 0	1 , 0	1 , 5

In this game $S^{\infty} = S_m^{\infty} = S$

while $S_s^{\infty} = \{(A,E),(A,F),(B,D),(C,D)\}$

Thus every strategy profile is compatible with *common belief* of rationality while only (A,E), (A,F), (B,D) and (C,D) are compatible with *common knowledge* of rationality

CREDITS

The link between the iterated deletion of strictly dominated strategies and the informal notion of common belief of rationality was first shown by Bernheim (1984) and Pearce (1984)

The first explicit epistemic characterization was provided by Tan and Werlang (1998) using a universal type space.

The state space formulation used in Propositions 5 and 6 is due to Stalnaker (1994), but it was implicit in Brandenburger and Dekel (1987).

Propositions 7 and 8 are due to Stalnaker (1994) (with a correction given in Bonanno and Nehring, 1996b).

To my knowledge, Propositions 1, 2, 3 and 4 have not been explicitly stated before.

References and further details can be found in

Battigalli, Pierpaolo and Bonanno Giacomo, “Recent results on belief, knowledge and the epistemic foundations of game theory”, *Research in Economics*, 53 (2), June 1999, pp. 149-225.

For a syntactic version of Propositions 1, 2, 3 and 4 see
Giacomo Bonanno, A syntactic approach to rationality in games, Working Paper, University of California, Davis (http://www.econ.ucdavis.edu/faculty/bonanno/PDF/CBR.pdf)