Ricerche Economiche (1994) **48**, 23–43

Information, rational beliefs and equilibrium refinements

Giacomo Bonanno

Department of Economics, University of California, Davis, CA 95616-8578, U.S.A.

(Received 1 April 1993, accepted for publication 26 July 1993)

Summary

Given an extensive game *G*, three subsets of the normal-form equivalence class of *G* are defined: the subset of simultaneous games [denoted by Sim(*G*)] the subset of subgame-preserving quasi-simultaneous games [denoted by SubSim(*G*)] and, finally, the subset consisting of the game *G* itself. We show that by applying the notion of rational profile of beliefs (which is formulated independently of the notion of strategy and therefore of Nash equilibrium) to the games in Sim(*G*) one obtains exactly the Nash equilibria of *G*, by applying it to the games in SubSim(*G*) one obtains exactly the subgame-perfect equilibria of *G* and, finally, by applying it to *G* itself one obtains a (strict) refinement of subgame-perfect equilibrium.

J.E.L. Classification: C72.
Keywords: Information, rational beliefs, equilibrium refinements.

1. Introduction

For the past twenty years or so one of the main research programs within non-cooperative game theory has been to refine the Nash equilibrium concept so as to better capture the notion of rationality in interactive situations. This research program has yielded a plethora of solution concepts (for a partial survey see Van Damme, 1987). Although none of these solution concepts has been derived from an axiomatic formulation of the notion of rationality, one view, that would probably be shared by many game-theorists, is that each refinement of Nash equilibrium incorporates a well-defined notion of rationality and that, furthermore, *what is involved in moving from one solution concept to a refinement of it is a strengthening of the underlying notion of rationality*. Thus, for example, according to this view, there is some notion of rationality from which one can deduce the solution concept "Nash equi-

0035–5054/94/010023 + 21 $08.00/0

librium". A strengthening of that notion of rationality yields the solution concept "subgame-perfect equilibrium" (Selten, 1965, 1975). A further strengthening of that notion of rationality yields "sequential equilibrium" (Kreps & Wilson, 1982), and so on.

In this paper we offer an alternative point of view, by showing that *it is possible to interpret equilibrium refinements as stemming from an increase in the amount of information conveyed to the players* (rather than from a strengthening of the notion of rationality). Furthermore, we provide an axiomatic definition of rationality, *which is formulated independently of the notion of strategy* (and, therefore, of Nash equilibrium).

Our approach builds on the concepts of information and belief introduced in Bonanno (1992*a*,*b*). Fix an extensive game and let Z be the set of terminal nodes. For every player i and for every node t, a subset $K_i(t)$ of Z is defined and is interpreted as the information received by player i when the play of the game reaches node t. Thus if, say, $K_2(t)=\{z_1, z_3, z_7\}$ then, when node t is reached, player 2 learns that the play of the game so far has been such that only terminal nodes z_1, z_3 or z_7 can be reached. A belief of player i is then defined as a function that associates with every node t an element of the set $K_i(t)$, denoted by $\beta_i(t)$. The interpretation is that if, say, $K_2(t)=\{z_1, z_3, z_7\}$ and $\beta_2(t)=z_3$, then, at node t, player 2 knows (is informed) that the play of the game can only end either at node z_1 or at z_3 or at z_7 and believes that the outcome will actually be z_3. A profile of beliefs is a list of beliefs, one for each player. Four consistency properties define the notion of rational profile of beliefs. It is shown in Bonanno (1992*b*) that the notion of rational profile of beliefs gives rise to a refinement of subgame-perfect equilibrium.

The question of if, and how, the amount of information conveyed to the players during the play of a game can have an effect on the solutions of the game can be approached from two different directions. One approach is to fix an extensive game and vary the "extended information function" $K_i(t)$ and show that by refining K one obtains a refinement of Nash equilibrium (this is the line of inquiry followed in Bonanno, 1992*d*). Another approach is to fix the extended information function $K_i(t)$ and consider different classes of extensive games within the same normal-form equivalence class of a given game. This is the approach followed in this paper. Given an extensive game G, we identify three subsets of the normal-form equivalence class of G: the subset of simultaneous games (denoted by Sim(G) and defined in Section 3), the subset of subgame-preserving simultaneous games (denoted by SubSim(G) and defined in Section 4) and, finally, the subset consisting of the game G itself. We show that (1) by applying the notion of rational profile of beliefs to the games in Sim(G) one obtains exactly the Nash equilibria of G,† (2) by applying it to the games in SubSim(G) one obtains exactly the subgame-perfect equilibria of G and, finally, (3) by applying the notion of rational profile of beliefs to G itself one obtains a (strict) refinement of the notion of subgame-perfect equilibrium.

These results are in a sense trivial, since they clearly hold for the standard sequential equilibrium concept and probably for some of the many notions of perfect Bayesian equilibrium that have been proposed. The reason why they are proved here in relation to the notion of rational profile of beliefs is two-fold. First of all, since the notion of rational profile of beliefs is based exclusively on the extended information function $K_i(t)$, and does not make any reference to the notion of strategy, it is not *a priori* obvious that, when applied to different subsets of the normal-form equivalence class of a given game, it should yield *exactly* the notions of Nash equilibrium, subgame-perfect equilibrium, etc. Secondly, precisely because rational profiles of beliefs are based on the extended information function $K_i(t)$, the difference in results can (indeed, it must) be interpreted in terms of a dependence of the solution on the amount of information conveyed to the players.‡

In order to make the exposition as simple and as transparent as possible, we shall limit ourselves to pure beliefs (see Section 2), pure strategies and games without chance moves. A very general analysis that allowed for mixed beliefs, mixed strategies and chance moves would require more complex definitions and would obscure the simple point of this paper, namely that refinements of Nash equilibrium may reflect, not a strengthening of the notion of rationality, but rather an increase in the amount of information conveyed to the players.

2. Preliminary definitions

We begin by reviewing the notation and some of the definitions of Bonanno (1992*a*,*b*). Fix a finite extensive game. Let X be the set of

† By "one obtains exactly" we mean that there is a one-to-one correspondence.

‡ The results of this paper also raise some doubts about a prominent view among game theorists, namely that there are no important details in the extensive form, other than those that are captured by the corresponding normal (or strategic) form. This view was put forward in a very influential paper by Kohlberg and Mertens (1986). (In a recent paper, Mailath, Samuelson and Swinkels (1993) have pointed out a subtle structural relationship between a normal form game and its many extensive-form representations.) Our results show that the extensive form may be important in that it specifies the amount of information that the players obtain during the play of the game. As this information becomes more refined, so do the implications of rationality as captured by the notion of rational profile of beliefs.

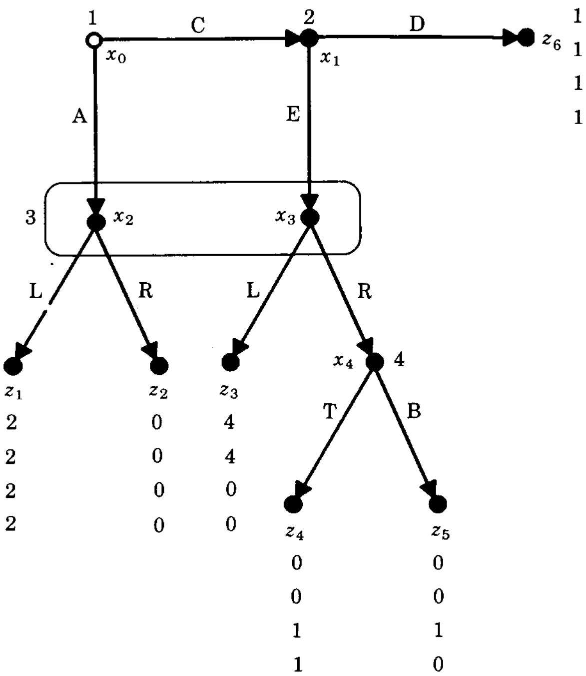

FIGURE A. An extensive game used to illustrate the notion of information.

decision nodes, Z the set of *terminal* nodes, and $T = X \cup Z$. For every $t \in T$, let $\theta(t) \subseteq Z$ be the set of terminal nodes that can be reached from t (for example, in the game of Figure A, $\theta(x_1) = \{z_3, z_4, z_5, z_6\}$). Clearly, for every $z \in Z$, $\theta(z) = \{z\}$.

We denote by x_0 the root of the tree and for every node $t \neq x_0$ we shall denote the immediate predecessor of t by p_t. Finally, for every node t and for every player i, $H_i(t)$ is the set of information sets that satisfy the following property: $h \in H_i(t)$ if and only if h is an information set of player i and there is a node $y \in h$ that is a successor of t.

The information received by player i when the play of the game reaches node t is denoted by $K_i(t)$. There are several "extended information" functions K: $I \times T \rightarrow 2^Z$ (where I is the finite set of players and 2^Z denotes the set of subsets of Z) that can be defined subject to some natural restrictions (e.g. that they be coherent with the information structure of the extensive game). We shall use the following function, which represents the maximum amount of information that can be conveyed to the players:†

† On this see Bonanno (1992*a*), in particular the discussion in the appendix. It should be noted, however, that the function $K(\cdot,\cdot)$ represents the maximum amount of information only if one considers the class of *all* the extensive games. On the other hand, if one restricts attention to a subclass of extensive games, then the function $K(\cdot,\cdot)$ can be refined. A characterization of the notion of maximum information for multistage games is given in Battigalli and Bonanno (1993).

(1) **[At the root of the tree all the players have the same information]:**

$$K_i(x_0) = Z, \text{ for every player } i.$$

(2) **[When a terminal node is reached, all the players are informed]:**

$$K_i(z) = \{z\}, \text{ for every } z \in Z \text{ and for every player } i.$$

(3) **[Coherence with the information structure]:** if x is a decision node that belongs to information set h of player i, then $K_i(x) = \bigcup_{y \in h} \theta(y)$.

(4) **[If x is a node such that, either there are no decision nodes of player i after x, or the information sets of player i that come after x consist of successors of x, then player i is told that node x has been reached]:** if x is a decision node of a player different from player i and either $H_i(x) = \emptyset$ (where $\emptyset$ denotes the empty set) or, for every $h \in H_i(x)$, $\bigcup_{y \in h} \theta(y) \subseteq \theta(x)$ then $K_i(x) = \theta(x)$.

(5) **[Players remember the choices they made]:** if x is a decision node of a player different from player i and the condition given under (4) is not satisfied† and x is an immediate successor of decision node t of player i and $\{(t_1, x_1), (t_2, x_2), \ldots, (t_m, x_m)\}$ is the choice of player i that leads from t to x,‡ then

$$K_i(x) = \theta(x_1) \cup \theta(x_2) \cup \ldots \cup \theta(x_m).$$

(6) **[In every other case, players receive no new information]:** finally, if x is a decision node of a player different from player i and it does not satisfy conditions (4) and (5), then $K_i(x) = K_i(p_x)$ (recall that p_x is the immediate predecessor of node x).

For example, in the game of Figure A we have:

By (1): $K_i(x_0) = Z = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ for all $i = 1, 2, 3, 4$.
By (2): $K_i(z_j) = \{z_j\}$ for all $i = 1,2,3,4$ and for all $j = 1, \ldots, 6$.
By (3): $K_2(x_1) = \theta(x_1) = \{z_3, z_4, z_5, z_6\}$.
By (4): $K_1(x_1) = K_4(x_1) = \theta(x_1) = \{z_3, z_4, z_5, z_6\}$.
By (6): $K_3(x_1) = K_3(x_0) = Z$.
By (4): $K_1(x_3) = K_2(x_3) = K_4(x_3) = \theta(x_3) = \{z_3, z_4, z_5\}$.

† That is, there exists an $h \in H_i(x)$ and a node $y \in h$ such that y is not a successor of x.

‡ Thus $t = t_j$ and $x = x_j$, for some $j = 1, \ldots, m$. Recall that a choice c at information set $h = \{t_1, \ldots, t_m\}$ is a set of arcs $c = \{(t_1, x_1), (t_2, x_2), \ldots, (t_m, x_m)\}$ where, for each $k = 1, \ldots, m$, node x_k is an immediate successor of node t_k.

By (3): $K_3(x_3)=\{z_1, z_2, z_3, z_4\ z_5\}$.
By (3): $K_4(x_4)=\theta(x_4)=\{z_4, z_5\}$.
By (4): $K_1(x_4)=K_2(x_4)=K_3(x_4)=\theta(x_4)=\{z_4, z_5\}$.
By (4): $K_1(x_2)=K_2(x_2)=K_4(x_2)=\theta(x_2)=\{z_1, z_2\}$.
By (3): $K_3(x_2)=\{z_1, z_2, z_3, z_4, z_5\}$.

REMARK 1: *by (3) it is clear that if h is an information set of player i, and x and y are two nodes in h, then $K_i(x)=K_i(y)$. Thus it makes sense to write $K_i(h)$ for player i's information at her information set h.*

In order to simplify the exposition, from now on we shall restrict attention to extensive games with perfect recall that have no chance moves.

DEFINITION 1: *a* (pure) belief *of player i is a function*

$$\beta_i\colon T\to Z$$

satisfying the following properties:
(i) $\beta_i(t)\in K_i(t)\quad \forall t\in T$,
(ii) *if x and y belong to information set h of player i, then $\beta_i(x)=\beta_i(y)$.*

Condition (i) says that what a player believes must be consistent with what he knows, and condition (ii) says that a player cannot have different beliefs at two nodes that belong to one of his information sets (since his information is the same at both nodes). Thus it makes sense to write $\beta_i(h)$ for player i's belief at her information set h.

DEFINITION 2: *a* profile of beliefs *is an n-tuple $\beta=(\beta_1,\ldots,\beta_n)$, where, for each player $i=1,\ldots,n$, β_i is a belief of player i.*

We shall make use of the following notation: given a decision node x, $\Sigma(x)$ denotes the set of immediate successors of x.

DEFINITION 3: *we say that a profile of beliefs β is* rational *if it satisfies the following properties (a discussion of these properties follows):*

(1) ***[Contraction Consistency]*** *If y is a successor of x† and $\beta_i(x)\in K_i(y)$, then $\beta_i(y)=\beta_i(x)$.*
(2) ***[Tree Consistency]*** *Let h be an information set of player i and let $x\in h$ be the predecessor of $\beta_i(h)$. Then*

† It is shown in Bonanno (1992*a*) that in a game with perfect recall the following is true for every player i and for every two nodes x and y: if y is a successor of x then $K_i(y)\subseteq K_i(x)$.

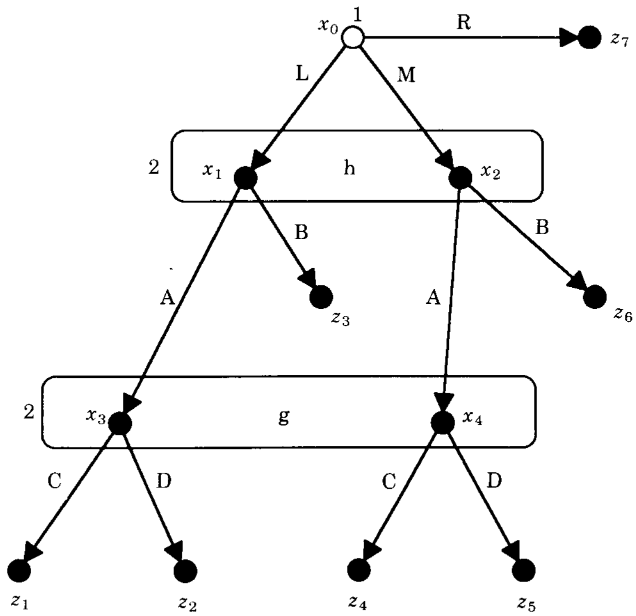

FIGURE B. Illustration of the implications of the property of Tree Consistency.

$$\beta_i(y) \in \theta(y), \qquad \forall y \in \Sigma(x).$$

(3) ***[Individual Rationality]*** *Let h be an information set of player i and let $x \in h$ be the predecessor of $\beta_i(h)$. Then*

$$U_i(\beta_i(h)) \geqslant U_i(\beta_i(y)) \qquad \forall y \in \Sigma(x),$$

where *U_i: $Z \rightarrow \mathcal{R}$ is player i's payoff function ($\mathcal{R}$ denotes the set of real numbers).*

(4) ***[Choice Consistency]*** *Let node x belong to information set h of player i, and let c be the choice at h that precedes $\beta_i(h)$. Then, for every player j, if $\beta_j(x)$ comes after choice d at h, it must be $d = c$.*

Property (1) says that a player does not change her belief unless she has to, that is unless her previous belief is inconsistent with the new information.

Tree consistency does not allow a player to change her opinion about her position in an information set after her own choice. In other words, its purpose is to rule out situations like the one illustrated in Figure B. There we have that $K_2(h) = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ where $h = \{x_1, x_2\}$ is the first information set of player 2, and $K_2(g) = \{z_1, z_2, z_4, z_5\}$ where $g = \{x_3, x_4\}$ is the second information set of player 2.

Suppose $\beta_2(h) = z_6$ and $\beta_2(g) = z_1$. This belief of player 2 is inconsistent because believing in z_6 at h means believing that node x_2 was reached. Given this belief, if player 2 takes action A, so that the play of the game proceeds to information set g, then node x_4 must be reached, and from x_4 terminal node z_1 cannot be reached. In this example, property (2) requires that if $\beta_2(h)=z_6$ then either $\beta_2(g)=z_4$ or $\beta_2(g)=z_5$.

The motivation for property (3) is as follows. If terminal node z represents what player i believes at her information set h [that is, if $z=\beta_i(h)$], then it means that player i believes that she is at that node x in h which lies on the play to z. Suppose that $U_i(z)<U_i(\beta_i(y))$ where y is an immediate successor of x. Then believing in z (at h) is irrational for player i because, instead of making the choice required to reach z, she can—according to her beliefs and by making another choice—move the game to node y where, again according to her beliefs, the game will evolve to an outcome which she prefers to z.

Property (4) says that if player j believes that the play of the game has reached player i's information set h, then player j's belief concerning the *choice* that will be made by i at h must be the same as the choice implied by i's belief at h (although i and j might disagree on the node at which this choice would be made). A justification for this property could be that player j puts himself in the shoes of player i and correctly predicts the choice that player i would make at her information set h.

Given an arbitrary profile of beliefs β we can extract from it a pure-strategy profile $\sigma=\xi(\beta)$ as follows: if h is an information set of player i and c is the choice at h that precedes $\beta_i(h)$, set $\sigma_i(h)=c$, that is, c is the choice selected (with probability 1) by player i's strategy at information set h.

The following result is proved in Bonanno (1992*b*).

PROPOSITION 1: *fix an extensive game G.*† *Let β be a rational profile of beliefs and let $\sigma=\xi(\beta)$ be the corresponding strategy profile. Then σ is a subgame-perfect equilibrium of G.*

3. Definition of Sim(*G*)

Given an extensive game G, we denote by $\mathscr{E}(G)$ the normal-form equivalence class of G, that is, the set of extensive games that have the same normal form as G. In this section we discuss a subset of this class, denoted by Sim(G).

Define an extensive game to be *simultaneous* if every play crosses all the information sets. Given an extensive game G, we denote by Sim(G) the games in $\mathscr{E}(G)$ that are simultaneous and satisfy the

† Recall that throughout the paper we restrict attention to extensive games with perfect recall that have no chance moves.

		Player 2: D	Player 2: E
Player 1	A	2,2,2,2	2,2,2,2
	C	1,1,1,1	4,4,0,0

Player 3 chooses L

		Player 2: D	Player 2: E
Player 1	A	0,0,0,0	0,0,0,0
	C	1,1,1,1	0,0,1,1

Player 3 chooses R

Player 4 chooses T

		Player 2: D	Player 2: E
Player 1	A	2,2,2,2	2,2,2,2
	C	1,1,1,1	4,4,0,0

Player 3 chooses L

		Player 2: D	Player 2: E
Player 1	A	0,0,0,0	0,0,0,0
	C	1,1,1,1	0,0,1,0

Player 3 chooses R

Player 4 chooses B

FIGURE C. The normal or strategic form of the extensive game of Figure A.

property that every player has exactly one information set (Sim(G) is non-empty and every element of it can be obtained from G by applying the transformations described by Thompson, 1952). For example, if G is the game of Figure A (whose normal form is shown in Figure C), then Figure D shows one element of Sim(G) (the others are obtained by changing the order in which players choose along any given play).

For later use we shall list the (pure-strategy) equilibria of the game of Figure A.

REMARK 2: *the game of Figure A has the following pure-strategy equilibria:*

(1) *Nash: (A,D,L,T), (A,D,L,B), (C,D,R,T) and (C,D,R,B).*
(2) *Subgame-perfect: (A,D,L,T), (C,D,R,T).*
(3) *Sequential: (C,D,R,T) with player 3 assigning probability 1*

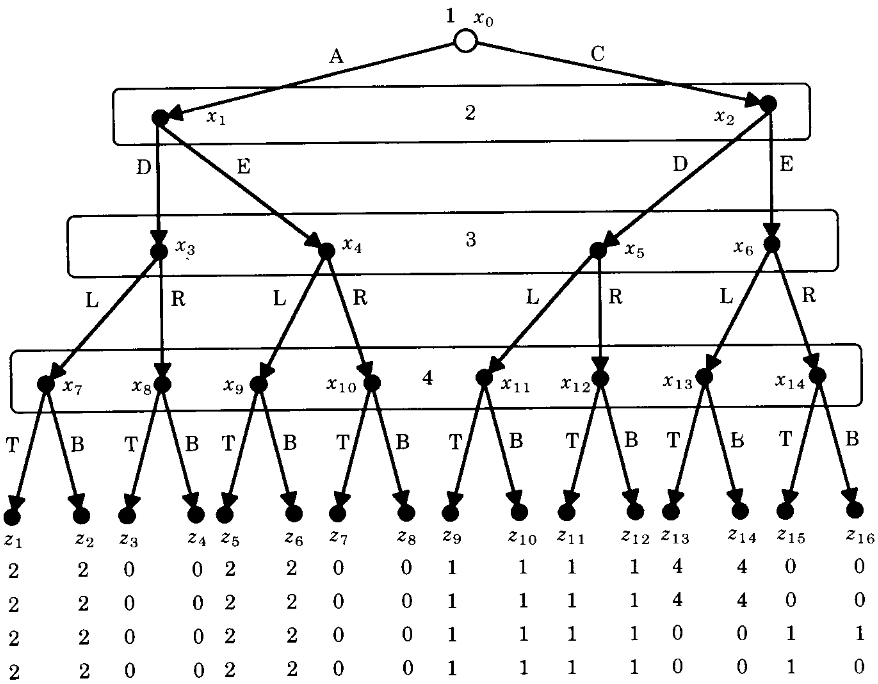

FIGURE D. A game in Sim(G) where G is the extensive game of Figure A.

> *to node x_3 (we are only considering simple assessments here, as defined in Section 5).*

LEMMA 1: *let G be an extensive game. Let G_0 be an arbitrary element of Sim(G) and let β be a rational profile of beliefs of G_0. Let $\sigma = \zeta(\beta)$ be the corresponding pure-strategy profile. Then σ is a Nash equilibrium of G_0.*

PROOF: it follows from proposition 1, since every subgame-perfect equilibrium is also a Nash equilibrium. ■

We now want to prove the converse of lemma 1, namely that given a pure-strategy Nash equilibrium σ of G_0 we can extract from it a rational profile of beliefs of G_0.

First a piece of notation: given a pure-strategy profile σ, for every node x we shall denote by $\zeta(x|\sigma)$ the terminal node reached from x by following the choices dictated by σ. Now, given a simultaneous game and a pure-strategy profile σ, let $\beta = \phi(\sigma)$ be the profile of beliefs obtained as follows. For every player i and for every node x,

(1) if x is a decision node of player i or a predecessor of a decision node of player i, set $\beta_i(x) = \zeta(x_0|\sigma)$,
(2) if x is a successor of a decision node of player i, set $\beta_i(x) = \zeta(x|\sigma)$.†

Remark 3: *it is easy to verify that if σ is a pure-strategy profile then $\xi(\phi(\sigma)) = \sigma$, that is, $\xi = \phi^{-1}$.*

Lemma 2: *let G be an extensive game. Let G_0 be an arbitrary element of Sim(G) and σ a pure-strategy Nash equilibrium of G_0. Let $\beta = \phi(\sigma)$ be the corresponding profile of beliefs of G_0. Then β is rational.*

Proof: first of all, note that, in a simultaneous game where every player has exactly one information set, for every player i we have that:
(1) if x is a decision node of player i or a predecessor of a decision node of player i, $K_i(x) = Z$,
(2) if x is a successor of a decision node of player i then $K_i(x) = \theta(x)$.

Thus if σ is a pure-strategy profile of G_0 and $\beta = \phi(\sigma)$, then β is indeed a profile of beliefs [that is, for every player i, it is true that $\beta_i(t) \in K_i(t)$ for every node t and that if x and y belong to the same information set of player i then $\beta_i(x) = \beta_i(y)$]. We need to show that β is rational.

By (2) of the definition of $\phi(\sigma)$, Tree Consistency is satisfied, since, for every node y that is a successor of a decision node of player i, $\beta_i(y) = \zeta(y|\sigma) \in \theta(y)$.

Let x belong to an information set of player i and let j be another player. Then, by definition of $\phi(\sigma)$, $\beta_i(x) = \zeta(x_0|\sigma)$ and either $\beta_j(x) = \zeta(x_0|\sigma)$ or $\beta_j(x) = \zeta(x|\sigma)$. Thus both $\beta_i(x)$ and $\beta_j(x)$ come after the same choice at h. Hence Choice Consistency is satisfied.

Now, fix an arbitrary player i and two nodes x and y such that y is a successor of x and $\beta_i(x) \in K_i(y)$. We want to show that $\beta_i(y) = \beta_i(x)$ (Contraction Consistency). If y is a decision node of player i or a predecessor of a decision node of player i, then $\beta_i(y) = \beta_i(x) = \zeta(x_0|\sigma)$. If x is a decision node of player i or a predecessor of a decision node of player i while y is a successor of a decision node of player i, then $\beta_i(x) = \zeta(x_0|\sigma)$ and $\beta_i(y) = \zeta(y|\sigma)$ and $K_i(y) = \theta(y)$. Since by hypothesis $\beta_i(x) \in K_i(y)$, the play to $\zeta(x_0|\sigma)$ must go through node y. Thus $\zeta(y|\sigma) = \zeta(x_0|\sigma)$. Hence $\beta_i(y) = \beta_i(x)$. Finally, if x is a successor of a decision node of player i, then $K_i(x) = \theta(x)$, $K_i(y) = \theta(y)$, $\beta_i(x) = \zeta(x|\sigma)$

† For example, consider the game of Figure D and the pure-strategy profile (A, E, L, T). Then $\beta = \phi(\sigma)$ is given as follows: $\beta_1(x_0) = \beta_1(x_1) = \beta_2(x_0) = \beta_2(x_1) = \beta_2(x_2) = \beta_3(x_0) = \beta_3(x_1) = \beta_3(x_2) = \beta_3(x_3) = \beta_3(x_4) = \beta_3(x_5) = \beta_3(x_6) = \beta_1(x_4) = \beta_2(x_4) = z_5$; $\beta_4(x_j) = z_5$ for all $j = 1, \ldots, 14$; $\beta_1(x_2) = \beta_1(x_6) = z_{13}$; $\beta_1(x_5) = \beta_2(x_5) = z_9$; $\beta_1(x_3) = \beta_2(x_3) = z_1$; and for $i = 1, 2, 3$ $\beta_i(x_7) = z_1$, $\beta_i(x_8) = z_3$, $\beta_i(x_9) = z_5$, $\beta_i(x_{10}) = z_7$, $\beta_i(x_{11}) = z_9$, $\beta_i(x_{12}) = z_{11}$, $\beta_i(x_{13}) = z_{13}$, $\beta_i(x_{14}) = z_{15}$.

and $\beta_i(y) = \zeta(y|\sigma)$. Since by hypothesis $\beta_i(x) \in K_i(y)$, the path from x to $\zeta(x|\sigma)$ must go through node y. Thus $\zeta(y|\sigma) = \zeta(x|\sigma)$. Hence $\beta_i(y) = \beta_i(x)$.

It only remains to show that, for each player i, β_i satisfies Individual Rationality. Suppose that there is a player i who violates Individual Rationality. Let h be the unique information set of player i, $x \in h$ be the node that belongs to the play to $\zeta(x_0|\sigma)$ [recall that, by definition of $\phi(\sigma)$, $\beta_i(h) = \zeta(x_0|\sigma)$] and y be an immediate successor of x such that $U_i(\beta_i(y)) > U_i(\zeta(x_0|\sigma))$. Since $\beta_i(y) = \zeta(y|\sigma)$, we have that

$$U_i(\zeta(y|\sigma)) > U_i(\zeta(x_0|\sigma)). \tag{1}$$

Let $\sigma_i(h) = c$ and let d be the choice at h that precedes y. It follows from (1) that $d \neq c$. Let $\sigma_i'(h) = d$ and $\sigma' = (\sigma_i', \sigma_{-i})$. Then $\zeta(x_0|\sigma') = \zeta(y|\sigma)$. Thus, by (1), by switching from σ_i to σ_i' player i can increase her payoff, contradicting the assumption that σ is a (pure-strategy) Nash equilibrium of G_0. ∎

We can now prove the main result of this section, namely that, given an arbitrary extensive G, by applying the notion of rational profile of beliefs to the games in Sim(G) one obtains exactly the Nash equilibria of G.

PROPOSITION 2: *let G be an extensive game and G_0 be an arbitrary element of Sim(G). Then there is a one-to-one correspondence between the set of pure-strategy Nash equilibria of G and the set of rational profiles of beliefs of G_0.*

PROOF: by lemmas 1 and 2 and remark 3, there is a one-to-one correspondence between the set of pure-strategy Nash equilibria of G_0 and the set of rational profiles of beliefs of G_0. Since G and G_0 have the same set of pure-strategy Nash equilibria, the proof is complete. ∎

For example, there is a one-to-one correspondence between the pure-strategy Nash equilibria of the game of Figure A (cf. Remark 2) and the rational profiles of beliefs of the game of Figure D.

4. Definition of SubSim(G)

Fix an extensive game G. In this section we consider another subset of $\mathscr{E}(G)$ (the normal-form equivalence class of G) that we will call the set of *subgame-preserving quasi-simultaneous games* and denote it by SubSim(G). In order to obtain an element G' of SubSim(G) from G we transform every subgame of G into an

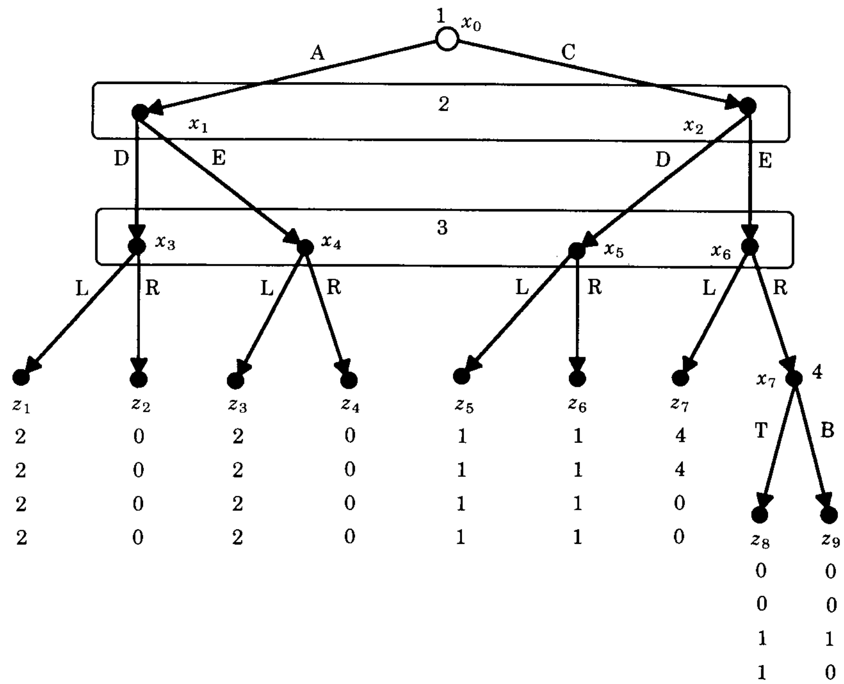

FIGURE E. A game in SubSim(G) where G is the extensive game of Figure A.

equivalent simultaneous version, but in such a way that G' has the same number of subgames as G. Before we proceed to the formal definition, we give an example. Let G be the game of Figure A. Then the game of Figure E is an element of SubSim(G).

In order to obtain an element of SubSim(G) from G apply the following algorithm:

STEP 1: find a smallest subgame of G (that is, a subgame of G that has no proper subgames), call it G_1. Replace G_1 with an arbitrary element of Sim(G_1), call it G_1'.

STEP 2: temporarily replace G_1'. with a terminal node z_1^{temp}. Denote the resulting game by G/G_1.

STEP 3: find a smallest subgame of G/G_1, call it G_2. Replace G_2 with an arbitrary element of Sim(G_2), call it G_2'.

STEP 4: temporarily replace G_2'. with a terminal node z_2^{temp}. Denote the resulting game by G/G_1G_2.

STEP 5: repeat until a game $G/G_1G_2\ldots G_m$ is obtained that has no proper subgames.

Step 6: replace $G/G_1G_2\ldots G_m$ with an arbitrary element of $\mathrm{Sim}(G/G_1G_2\ldots G_m)$.

Step 7: replace z_j^{temp} with G_j' for all $j=m, m-1,\ldots,2,1$ (in this order).

Lemma 3: *let G be an extensive game and G_0 be an arbitrary element of SubSim(G). Let β be a rational profile of beliefs of G_0 and $\sigma=\xi(\beta)$. Then σ is a subgame-perfect equilibrium of G_0.*

Proof: it follows from proposition 1. ■

Now we want to prove the converse, namely that given a subgame-perfect equilibrium of G_0 we can extract from it a rational profile of beliefs.

First of all, given an extensive game, define a node x to be a *root-node* if it is the root of a subgame. For every node x, define $\rho(x)$ as follows: $\rho(x)=x$ if x is a root-node, otherwise $\rho(x)$ is the closest predecessor of x that is a root-node. Now, given an element $G_0\in\mathrm{SubSim}(G)$ and a pure-strategy profile σ of G_0, let $\beta=\phi(\sigma)$ be obtained as follows. For every player i and for every node x,

(1) if x is a decision node of player i, for every node t on the path from $\rho(x)$ to x (including $\rho(x)$ and x) set $\beta_i(t)=\zeta(\rho(x)|\sigma)$,
(2) for every other node x, set $\beta_i(x)=\zeta(x|\sigma)$ [that is, $\beta_i(x)=\zeta(x|\sigma)$ if either no successor of x is a decision node of player i or if between x and every successor of x that is a decision node of player i there is a root-node].

[Note that if $G_0\in\mathrm{Sim}(G)$, then the closest predecessor of x that is a root-node is x_0. Hence, when applied to simultaneous games, the function ϕ defined here coincides with the function ϕ defined in Section 3. This fact justifies the use of the same symbol ϕ for the two functions.]†

Remark 4: *note that, as before, for every pure-strategy profile σ, $\xi(\phi(\sigma))=\sigma$, that is, $\xi=\phi^{-1}$.*

Lemma 4: *let G be an extensive game and G_0 an arbitrary element of SubSim(G). Let σ be a pure strategy subgame-perfect equilibrium of G_0 and $\beta=\phi(\sigma)$ be the corresponding profile of beliefs. Then β is rational.*

Proof: first of all, note that if $G_0\in\mathrm{SubSim}(G)$ then it has the

† For example, consider the game of Figure E and the pure-strategy profile (C,D,R,B). Then $\beta=\phi(\sigma)$ is given as follows: $\beta_1(x_0)=\beta_2(x_0)=\beta_2(x_1)=\beta_2(x_2)=\beta_1(x_2)=\beta_4(x_2)=\beta_1(x_5)=\beta_2(x_5)=\beta_4(x_5)=z_6$; $\beta_3(x_j)=z_6$ for $j=0,1,\ldots,6$; $\beta_1(x_1)=\beta_4(x_1)=\beta_1(x_3)=\beta_2(x_3)=\beta_4(x_3)=z_2$; $\beta_1(x_4)=\beta_2(x_4)=\beta_4(x_4)=z_4$; $\beta_1(x_6)=\beta_2(x_6)=\beta_4(x_6)=z_9$; $\beta_i(x_7)=z_9$ for all $i=1,2,3,4$.

following structure: if G_0' is a subgame of G_0 and G_0'' is the game obtained from G_0' by replacing all the proper subgames of G_0' by terminal nodes, then every player has at most one information set in G_0'' [thus if a player has two information sets in a subgame G_0' of G_0 then there is a proper subgame of G_0' to which one and only one of the two information sets belongs]; furthermore, all the information sets of G_0'' are crossed by all the paths from the root of G_0'' to terminal nodes. It follows that, for every player i, the following is true for G_0:

> if x is a decision node of player i, then $K_i(x) = \theta(\rho(x))$; furthermore, for every node y that lies on the path from $\rho(x)$ to x, $K_i(y) = \theta(\rho(x))$. For every other node t, $K_i(t) = \theta(t)$.

[In other words, let t be an arbitrary node. If there is no successor of t that is a decision node of player i, then $K_i(t) = \theta(t)$. Otherwise, let x be the closest successor of node t that is a decision node of player i; if t lies on the path from $\rho(x)$ to x, then $K_i(t) = \theta(\rho(x))$, otherwise $K_i(t) = \theta(t)$.]

The proof of this lemma is similar to that of lemma 2. First of all, it is clear that if σ is a pure-strategy profile of $G_0 \in \text{SubSim}(G)$ and $\beta = \phi(\sigma)$, then β is indeed a profile of beliefs [that is, for every player i, it is true that $\beta_i(t) \in K_i(t)$ for every node t and that if x and y belong to the same information set of player i then $\beta_i(x) = \beta_i(y)$]. We need to show that β is rational.

Tree Consistency follows from the fact that if y is an immediate successor of a decision node of player i, then $K_i(y) = \theta(y)$ (this is a consequence of the structure of G_0 described above: if a player has two information sets in a subgame G_0' of G_0 then there is a proper subgame of G_0' to which one and only one of the two information sets belongs).

Choice Consistency follows from the fact that for every two players i and j, and every node t, $\beta_i(t) = \zeta(x|\sigma)$ for some node x and $\beta_j(t) = \zeta(y|\sigma)$ for some node y. Thus if t belongs to information set h of player i and $\beta_j(t)$ comes after h, then both $\beta_i(t)$ and $\beta_j(t)$ come after the same choice at h, namely the choice prescribed by σ_i.

We now prove Contraction Consistency. Fix an arbitrary player i and two nodes x and y such that y is a successor of x and $\beta_i(x) \in K_i(y)$. We want to show that $\beta_i(y) = \beta_i(x)$. First of all, note that, by definition of $\phi(\sigma)$, $\beta_i(x) = \zeta(t|\sigma)$ for some node t that is either x itself or a predecessor of x. If $K_i(y) = \theta(y)$, then, since by hypothesis $\beta_i(x) \in K_i(y)$, it follows that the path from t to $\zeta(t|\sigma)$ goes through node y. Thus $\zeta(t|\sigma) = \zeta(y|\sigma)$. But $K_i(y) = \theta(y)$ implies that $\beta_i(y) = \zeta(y|\sigma)$. If $K_i(y) \neq \theta(y)$, then $K_i(y) = \theta(\rho(y))$. If x lies between $\rho(y)$ and y, then $\beta_i(y) = \beta_i(x) = \zeta(\rho(y)|\sigma)$. If x is a predecessor of $\rho(y)$, then from the hypothesis that $\beta_i(x) \in K_i(y)$ it follows that the path from t to $\zeta(t|\sigma)$ goes through $\rho(y)$. Hence $\zeta(t|\sigma) = \zeta(\rho(y)|\sigma)$.

It only remains to show that, for each player i, β_i satisfies Individual Rationality. Suppose that there is a player i who violates Individual Rationality. Then there is an information set h of player i such that, if $x \in h$ is the node that belongs to the path from $\rho(x)$ to $\zeta(\rho(x)|\sigma)$ (recall that, by definition of $\phi(\sigma)$, $\beta_i(h) = \zeta(\rho(x)|\sigma)$ where $\rho(x)$ is the closest predecessor of x that is a root-node) there is an immediate successor y of x such that $U_i(\beta_i(y)) > U_i(\zeta(\rho(x)|\sigma))$. It was shown above (cf. the proof of Tree Consistency) that $K_i(y) = \theta(y)$, implying that $\beta_i(y) = \zeta(y|\sigma)$. Thus it must be

$$U_i(\zeta(y|\sigma)) > U_i(\zeta(\rho(x)|\sigma)). \tag{2}$$

Let $\sigma_i(h) = c$ and let d be the choice at h that precedes y. It follows from (2) that $d \neq c$. Let σ'_i be the strategy obtained from σ_i by replacing $\sigma_i(h) = c$ with $\sigma_i(h) = d$ and let $\sigma' = (\sigma'_i, \sigma_{-i})$. Then $\zeta(\rho(x)|\sigma') = \zeta(y|\sigma)$. Thus, by (2), by switching from σ_i to σ'_i player i can increase his payoff in the subgame with root $\rho(x)$, contradicting the assumption that σ is a subgame-perfect equilibrium of G_0. ■

We can now prove the main result of this section, namely that, given an arbitrary extensive G, by applying the notion of rational profile of beliefs to the games in SubSim(G) one obtains exactly the subgame-perfect equilibria of G.

PROPOSITION 3: *let G be an extensive game and G_0 be an arbitrary element of SubSim(G). Then there is a one-to-one correspondence between the set of pure-strategy subgame-perfect equilibria of G and the set of rational profiles of beliefs of G_0.*

PROOF: by lemmas 3 and 4 and remark 4 there is a one-to-one correspondence between the set of pure-strategy subgame-perfect equilibria of G_0 and the set of rational profiles of beliefs of G_0. Since G and G_0 have the same pure-strategy subgame-perfect equilibria, the proof is complete. ■

REMARK 4: in order to fully appreciate the content of proposition 3, it is worth noting the following, which was pointed out by Battigalli and Li Calzi (1993: p. 96). There exist extensive-form solution concepts that satisfy the property of sequential rationality (which is also satisfied by rational profiles of beliefs: see proposition 4 below) and yet do not yield (or refine) subgame-perfection. An example can be found in the notion of "weak sequential equilibrium" (see Hillas, 1987; see also Myerson, 1991: pp. 170–175).†

† On the other hand, some extensive-form solution concepts do not satisfy subgame perfection because they purposefully weaken the sequential rationality condition: see, for example, Reny (1992).

5. Perfect Bayesian equilibria

In order to extend the analysis beyond subgame-perfect equilibria it is convenient to switch from strategy profiles to assessments, as defined by Kreps and Wilson (1982). Recall that an *assessment* is a pair (σ, υ), where σ is a strategy-profile and $\upsilon : T \to [0, 1]$ is a function (called a "system of beliefs" by Kreps and Wilson) satisfying the property that, for every information set h, $\Sigma_{x \in h}\, \upsilon(x) = 1$. We shall restrict attention to simple assessments. An assessment (σ, υ) is *simple* if σ is a pure-strategy profile and υ satisfies the following property: for every node x, either $\upsilon(x) = 0$ or $\upsilon(x) = 1$.

Given a (not necessarily rational) profile of beliefs β we can associate with it a simple assessment (σ, υ) by letting $\sigma = \xi(\beta)$ and $\upsilon = \tau(\beta)$, where $\tau(\beta)$ is defined as follows: if h is an information set of player i and $x \in h$ is the predecessor of $\beta_i(h)$, then $\upsilon(x) = 1$, and $\upsilon(y) = 0$ for all $y \in h / \{x\}$.

Definition 4: *a simple assessment (σ, υ) of an extensive game G is a* perfect Bayesian equilibrium *of G if there exists a profile of beliefs β of G such that β is rational and $(\sigma, \upsilon) = (\xi(\beta), \tau(\beta))$.*

Proposition 4: *let G be an extensive game and let the simple assessment (σ, υ) be a perfect Bayesian equilibrium of G. Then σ is a subgame-perfect equilibrium of G. Furthermore, (σ, υ) is sequentially rational.*

Proof: that σ is a subgame-perfect equilibrium of G follows directly from proposition 1, since $\sigma = \xi(\beta)$. It only remains to prove that (σ, υ) is sequentially rational. In order to do this we need the following lemma, which is proved in Bonanno (1992*b*; appendix B).

Lemma 5: *let G be an extensive game with perfect recall. Let β be a profile of beliefs of G that satisfies the properties of Contraction Consistency and Choice Consistency. Then for every player i and for every node x, if $\beta_i(x) \in \theta(x)$ then $\beta_i(x) = \zeta(x|\sigma)$ where $\sigma = \xi(\beta)$.*

Now we can prove that if β is a rational profile of beliefs of G and $(\sigma, \upsilon) = (\xi(\beta), \tau(\beta))$ is the corresponding simple assessment, then (σ, υ) is sequentially rational.

A simple assessment (σ, υ) is sequentially rational if it satisfies the following property. Fix an arbitrary information set h and let i be the corresponding player. Let $\hat{x} \in h$ be the node such that $\upsilon(\hat{x}) = 1$. Then

$$U_i(\zeta(\hat{x}|\sigma)) \geqslant U_i(\zeta(y|\sigma)) \qquad \text{for all } y \in \Sigma(\hat{x}) \tag{3}$$

(recall that $\Sigma(\hat{x})$ denotes the set of immediate successors of $\hat{x}$), that is, by switching to a different choice at information set h—given the belief that node $\hat{x}$ was reached with probability 1 and that future play will be according to σ—player i cannot increase her payoff.

By definition of $\tau(\beta)$, $\hat{x}$ is the node in h that precedes $\beta_i(h)$. Hence $\beta_i(h) \in \theta(\hat{x})$ and, by lemma 5,

$$\beta_i(h) = \zeta(\hat{x}|\sigma). \tag{4}$$

By Tree Consistency, for every $y \in \Sigma(\hat{x})$, $\beta_i(y) \in \theta(y)$. Hence by lemma 5,

$$\beta_i(y) = \zeta(y|\sigma) \text{ for every } y \in \Sigma(\hat{x}). \tag{5}$$

By Individual Rationality,

$$U_i(\beta_i(h)) \geq U_i(\beta_i(y)) \qquad \text{for every } y \in \Sigma(\hat{x}). \tag{6}$$

Putting together (4)–(6), we obtain (3). ■

Proposition 4 implies that the notion of perfect Bayesian equilibrium is stronger than that of subgame-perfect equilibrium, since the former satisfies sequential rationality, while the latter does not. For example, if G is the game of Figure A, then the only perfect Bayesian equilibrium is given by (C, D, R, T) with player 3 assigning probability 1 to node x_3 [this is also the only (simple) sequential equilibrium: see remark 2]. This can be shown by a minor adaptation of the argument given in Bonanno (1992*b*; section 4).

The following example, on the other hand, shows that not every perfect Bayesian equilibrium is a sequential equilibrium.† Consider the game of Figure F.

Let β be the following profile of beliefs:

$\beta_1(x_0) = z_{11}$, $\beta_1(x_1) = z_1$, $\beta_1(x_3) = z_2$, $\beta_1(x_4) = z_4$, $\beta_1(x_2) = z_{10}$, $\beta_1(x_5) = z_7$, $\beta_1(x_6) = z_9$;
$\beta_2(x_0) = z_{11}$, $\beta_2(x_1) = \beta_2(x_2) = z_{10}$, $\beta_2(x_3) = z_2$, $\beta_2(x_4) = z_4$, $\beta_2(x_5) = z_7$, $\beta_2(x_6) = z_9$;
$\beta_3(x_0) = z_{11}$, $\beta_3(x_1) = z_1$, $\beta_3(x_3) = \beta_3(x_4) = z_2$, $\beta_3(x_2) = z_{10}$, $\beta_3(x_5) = z_7$, $\beta_3(x_6) = z_9$;
$\beta_4(x_0) = z_{11}$, $\beta_4(x_1) = z_1$, $\beta_4(x_3) = z_2$, $\beta_4(x_4) = z_4$, $\beta_4(x_2) = z_{10}$, $\beta_4(x_5) = \beta_4(x_6) = z_9$.

† For a more complete analysis of the relationship between the notion of sequential equilibrium and that of rational profile of beliefs, see Bonanno (1992c). In particular, it is shown there that the former is a refinement of the latter.

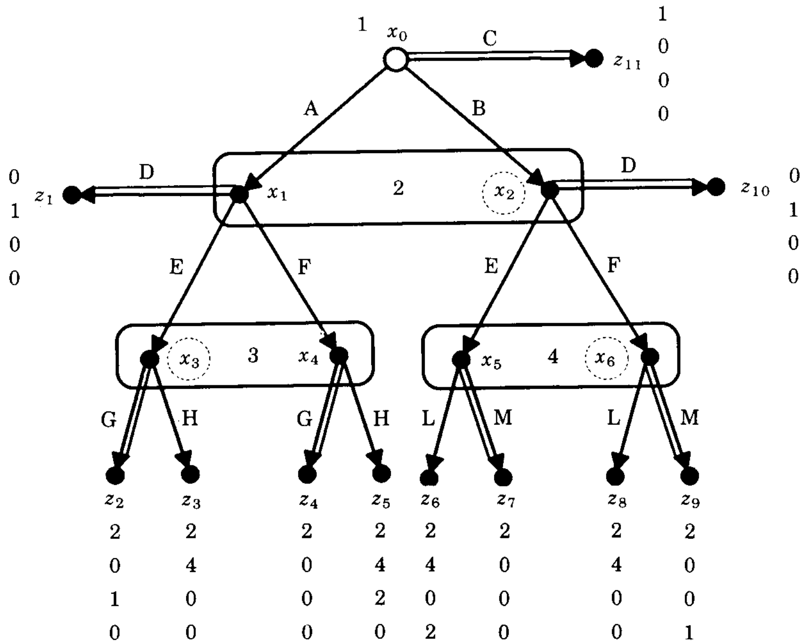

FIGURE F. An extensive game that shows that not every perfect Bayesian equilibrium is a sequential equilibrium.

It is easy to check that β is a rational profile of beliefs. Now, if $(\sigma, \upsilon) = (\xi(\beta), \tau(\beta))$, then $\sigma = (\mathrm{C,D,G,M})$ and $\upsilon(x_2) = 1$, $\upsilon(x_3) = 1$, $\upsilon(x_6) = 1$ (σ is denoted by double arcs in Figure F and υ by enclosing in a dotted circle the nodes that are assigned probability 1). This assessment is not a sequential equilibrium, because it does not satisfy the property of consistency as defined by Kreps and Wilson (1982). In fact, $\upsilon(x_3) = 1$ requires that along the sequence that converges to σ, E be assigned a probability of lower order than F, for example, $P\{E\} = \frac{1}{m}$ and $P\{F\} = \frac{1}{m^2}$, while $\upsilon(x_6) = 1$ requires the opposite. Intuitively, the property of consistency requires that there be agreement among all the players concerning the relative likelihood of any two deviations from the equilibrium strategies. Thus, if the equilibrium strategies are (C,D,G,M), then players 3 and 4 (whose information sets ought not to be reached) should agree—if asked to move—on whether deviation E is more or less likely than deviation F. Since player 3 assigns probability 1 to node x_3, he believes that E is infinitely more likely than F. On the other hand, player 4, by assigning probability 1 to node x_6, believes the opposite. Such disagreement is not allowed by the notion of consistency as defined by Kreps and Wilson. Finally, it is worth noting that in the game of Figure F there is no sequential equilibrium where player 1 chooses C with probability 1.

6. Conclusion

Given an extensive game, we associated with every node t and every player i a subset $K_i(t)$ of the set of terminal nodes, interpreted as player i's information when the play of the game reaches node t. A belief of player i was then defined as a map from the set of all nodes into the set of terminal nodes satisfying two main properties: what a player believes must be consistent with what she knows, and a player's beliefs must be the same at any two nodes that belong to one of her information sets (since her information is the same at those two nodes). Four natural properties (Contraction Consistency, Tree Consistency, Individual Rationality and Choice Consistency) were used to define the notion of rational profile of beliefs. Given an extensive game G we identified three subsets of the normal-form equivalence class of G: Sim(G), SubSim(G) and $\{G\}$. It was shown that: (1) the Nash equilibria of G are in one-to-one correspondence with the rational profiles of beliefs of an arbitrary element of Sim(G), (2) the subgame-perfect equilibria of G are in one-to-one correspondence with the rational profiles of beliefs of an arbitrary element of SubSim(G), and (3) the rational profiles of beliefs of G give rise to a (strict) refinement of the notion subgame-perfect equilibrium.

Acknowledgements

The author is grateful to an anonymous referee for very useful and constructive comments.

References

Battigalli, P. & Bonanno, G. (1993). *Synchronic information and common knowledge in extensive games.* Working Paper No. 93-11, Department of Economics, University of California, Davis, CA, U.S.A.

Battigalli, P. & Li Calzi, M. (1993). Four textbooks on game theory. *Ricerche Economiche*, **47**, 93–103.

Bonanno, G. (1992*a*). Players' information in extensive games. *Mathematical Social Sciences*, **24**, 35–48.

Bonanno, G. (1992*b*). Rational beliefs in extensive games. *Theory and Decision*, **33**, 153–176.

Bonanno, G. (1992*c*). *A characterization of sequential equilibrium.* Working paper No. 409, Department of Economics, University of California, Davis, CA, U.S.A.

Bonanno, G. (1992*d*). *Nash versus subgame-perfect equilibrium: the role of information.* Working paper No. 417, Department of Economics, University of California, Davis, CA, U.S.A.

Hillas, J. (1987). *Sequential equilibria and stable sets of beliefs.* I.M.S.S.S. Technical Report No. 518, Stanford University, CA, U.S.A.

Kohlberg, E. & Mertens, J.-F. (1986). On the strategic stability of equilibria. *Econometrica*, **54**, 1003–1037.

Kreps, D. & Wilson, R. (1982). Sequential equilibria. *Econometrica*, **50(4)**, 863–894.

Mailath, G., Samuelson, L. & Swinkels, J. (1993). Extensive form reasoning in normal form games. *Econometrica*, **61**, 273–302.

Myerson, R. (1991). *Game theory: analysis of conflict*. Cambridge, MA: Harvard University Press.

Reny, P. (1992). Backward induction, normal form perfection and explicable equilibria. *Econometrica*, **60**, 627–650.

Selten, R. (1965). Spieltheoretische Behandlung eines Oligopolmodells mit Nachfraegetragheit. *Zeitschrift für die gesamte Staatswissenschaft*, **12**, 301–324, 667–689.

Selten, R. (1975). Re-examination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory*, **4**, 25–55.

Thompson, F. (1952). *Equivalence of games in extensive form*. RM 759, The Rand Corporation.

Van Damme, E. (1987). *Stability and perfection of Nash equilibria*. New York: Springer-Verlag.