ON THE LOGIC OF COMMON BELIEF

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Abstract

We investigate an axiomatization of the notion of common belief (knowledge) that makes use of no rules of inference (apart from Modus Ponens and Necessitation) and highlight the property of the set of accessibility relations that characterizes each axiom.

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1. Introduction

Since Lewis’s (1969) and Aumann’s (1976) pioneering contributions, the notions of common knowledge and common belief have been investigated thoroughly, both semantically\(^1\) and syntactically\(^2\). Informally a proposition is common belief (knowledge) if everybody believes (knows) it, everybody believes (knows) that everybody believes (knows) it, and so on \textit{ad infinitum}. From a semantic point of view there are no difficulties in capturing the informal notion, since the intersection of an infinite family of sets is a meaningful concept (semantically, the notion of common belief is captured by the transitive closure of the union of the individual accessibility relations.). From a syntactic point of view, however, the informal notion cannot be captured directly because in a finitary logic formulae are required to be of finite length and, therefore, the conjunction of an infinite number of formulae is not itself a formula. Several axiomatizations of the notion of common belief (knowledge) have been offered (see Halpern and Moses, 1992, Lismont, 1992, 1993 and – for a recent survey – Lismont and Mongin, 1994). All of them include the so called “fixed-point” axiom

\[ \star A \rightarrow \Box (A \land \star A) \]

[where the intended interpretation of \( \star A \) is “it is common belief (knowledge) that \( A \)” and that of \( \Box A \) is “everybody believes (knows) that \( A \)”] together with some appropriate rule of inference. Halpern and Moses (1992) use the rule

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\(^1\) See, for example, Bacharach (1985), Binmore and Brandenburger (1990), Colombetti (1993), Geanakoplos (1992), Milgrom (1981), Samet (1990) and Shin (1993).

\[
A \rightarrow \square (A \land B), \\
A \rightarrow \diamondsuit B,
\]

while Lismont (1993) uses the rule

\[
A \rightarrow \square A \\
\square A \rightarrow \diamondsuit A.
\]

The purpose of this paper is to investigate an axiomatization of common belief (knowledge) that makes use of no rules of inference (apart from Modus Ponens and Necessitation) and to highlight the property of the set of accessibility relations that characterizes each axiom.

2. The formal system $K_{n^*}$ and its semantics

We consider a normal system with $(n+1)$ modal operators $\square, \lhd, \boxdot, \mathfrak{m}, \mathfrak{s}$. The intended interpretation of $\boxdot A$ (for $i = 1,...,n$) is “individual $i$ believes that $A$” whereas $\mathfrak{s} A$ is interpreted as “it is common belief that $A$”. The alphabet of the language consists of:

1. a countable set $S = \{ p_0, p_1, p_2, \ldots \}$ of sentence letters,
2. the connectives $\neg, \lor, \square, \ldots, \boxdot, \mathfrak{s}$ (where $n \geq 1$ is a natural number) and
3. the bracket symbols ( and ).

A word is a finite string of elements of the alphabet. The set $\mathcal{F}$ of formulae (or sentences) is the subset of the set of words defined recursively as follows:

1. for every sentence letter $p$, $(p) \in \mathcal{F},$

2. if $A \in \mathcal{F}$ then $\neg A \in \mathcal{F}$, $(\boxdot A) \in \mathcal{F}$, and, for every $i = 1, \ldots, n$, $(\boxdot A) \in \mathcal{F},$
(3) if $A, B \in \mathcal{F}$ then $(A \lor B) \in \mathcal{F}$. ³

We denote by $K_{\mathbf{pl}}$ the system or calculus specified by the following axiom schemata and rules of inference:

(1) All the tautologies (that is, a suitable axiomatization of Propositional Calculus),

(2) the schema $K$ (cf. Chellas, 1980):

\[
K_i (\Box \rightarrow B) \rightarrow (\Box_i \rightarrow \Box B), \quad \text{for every } i \in \{1, ..., n, \ast\},
\]

(3) the rule of inference $\text{Modus Ponens}$:

\[
\text{MP.} \quad \frac{A, A \rightarrow B}{B}
\]

(5) the rule of inference $\text{Necessitation}$:

\[
\text{RN.} \quad \frac{A}{\Box_i A} \quad \text{for every } i \in \{1, ..., n, \ast\}.
\]

We now turn to the semantics. A $\textit{standard frame}$ is an $(n+2)$-tuple $\langle W, R_1, ..., R_n, R_\ast \rangle$ where $W$ is a non-empty set whose members are called “worlds” and, for $i \in \{1, ..., n, \ast\}$, $R_i$ is a (possibly empty) binary “accessibility” relation on $W$. A $\textit{standard model}$ is an $(n+3)$-tuple $M = \langle W, R_1, ..., R_n, R_\ast, F \rangle$, where $\langle W, R_1, ..., R_n, R_\ast \rangle$ is a standard frame and $F : S \rightarrow 2^W$ is a function from the set of sentence letters $S$ into the set of subsets of $W$. We say that $M$ is $\textit{based on}$ the frame $\langle W, R_1, ..., R_n, R_\ast \rangle$.

³ We use the following metalinguistic abbreviations: $A \land B$ for $\neg (\neg A \lor \neg B)$ and $A \rightarrow B$ for $(\neg A) \lor B$. 

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Given a formula A and a standard model \( M = \langle W, R_1, ..., R_n, R_*, F \rangle \), the **truth set** of A in M, denoted by \( \models^M A \) is defined recursively as follows:

1. If \( A = (p) \) where p is a sentence letter, then \( \models^M A = F(p) \),
2. \( \models^M \neg A = W - \models^M A \) (that is, \( \models^M \neg A \) is the complement of \( \models^M A \))
3. \( \models^M A \lor B = \models^M A \cup \models^M B \)
4. For all \( i \in \{1, ..., n, *\} \)
   \[ \models^M \Box_i A = \{ \alpha \in W : \text{for all } \beta \text{ such that } \alpha R_i \beta, \beta \in \models^M A \} \].

If \( \alpha \in \models^M A \) we say that \( A \) is **true at world \( \alpha \) in model \( M \)\). An alternative notation for \( \alpha \in \models^M A \) is \( \models^M_\alpha A \) and an alternative notation for \( \alpha \notin \models^M A \) is \( \not\models^M_\alpha A \).

A formula A is **valid in model \( M = \langle W, R_1, ..., R_n, R_*, F \rangle \) if and only if \( \models^M_\alpha A \) for all \( \alpha \in W \).

The following proposition is a straightforward extension of a well-known result in modal logic (for a proof see Halpern and Moses, 1992).

**PROPOSITION 1.** The system \( K_{n*} \) is sound and complete with respect to the class of standard models, that is,

1. **Soundness:** every theorem of \( K_{n*} \) is valid in every standard model.
2. **Completeness:** if A is a formula that is valid in every standard model, then A is a theorem of \( K_{n*} \).
3. The logic of common belief

We shall consider the following axiom schemata, where $i \in \{1, \ldots, n\}$:

$S_i:\quad \Box A \rightarrow \Box_{i} A,$

$P_i:\quad \Box A \rightarrow \Box_{i} \Box A,$

$L:\quad \Box (A \rightarrow \Box_{1} A \land \ldots \land \Box_{n} A) \rightarrow \left(\Box_{1} A \land \ldots \land \Box_{n} A \rightarrow \Box A\right).$

$S_i$ says that if it is common belief that $A$, then individual $i$ believes that $A$. $P_i$ says that if it is common belief that $A$, then individual $i$ believes that it is common belief that $A$. Finally, $L$ says that if it is common belief that if $A$ then everybody believes that $A$, then if everybody believes that $A$ then it is common belief that $A$.

We say that a property $\mathcal{P}$ of the set $\{R_1, \ldots, R_n, R_\ast\}$ of accessibility relations characterizes axiom schema $\mathcal{A}$ if: (1) every instance of $\mathcal{A}$ is valid in every model that satisfies $\mathcal{P}$ and (2) given a frame that does not satisfy $\mathcal{P}$ there exists a model based on that frame and an instance of $\mathcal{A}$ which is not valid in that model.

PROPOSITION 2. In the following by ‘property’ we mean ‘property of the set $\{R_1, \ldots, R_n, R_\ast\}$ of accessibility relations’:

(i) Axiom schema $S_i$ is characterized by the following property:

\[ \forall \alpha, \beta \in W, \text{ if } \alpha R_i \beta \text{ then } \alpha R_\ast \beta; \]

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4 The name $S$ stands for Shared belief, $P$ for Public belief and the name $L$ was chosen because the corresponding axiom schema was first mentioned by Lismont (1993, p. 120, footnote 6).
(ii) Axiom schema \( P_i \) is characterized by the following property:\(^5\)

\[ \forall \alpha, \beta, \gamma \in W, \text{ if } \alpha R_i \beta \text{ and } \beta R_* \gamma \text{ then } \alpha R_* \gamma; \]

(iii) Axiom schema \( L \) is characterized by the following property:

\[ \forall \alpha, \beta \in W, \text{ if } \alpha R_* \beta \text{ then there exists a sequence } \langle \delta_1, ..., \delta_m \rangle \text{ in } W \]
(with \( m \geq 2 \)) and a sequence \( \langle i_1, ..., i_{m-1} \rangle \in \{1, ..., n\} \) such that:

1. \( \delta_1 = \alpha, \)
2. \( \delta_m = \beta, \)
3. for every \( k = 2, ..., m, \alpha R_* \delta_k, \) and
4. for every \( k = 1, ..., m-1, \delta_k R_k \delta_{k+1} \)

(that is, if \( \alpha R_* \beta \) then there is an \( R \)-path from \( \alpha \) to \( \beta \) – where \( R = R_1 \cup ... \cup R_n \) – such that for every node \( \gamma \) on this path, except possibly \( \alpha, \alpha R_* \gamma \).

**Proof.** The proof of (i) and (ii) is trivial and we omit it. As for (iii), let \( \langle W, R_1, ..., R_n, R_* \rangle \) be a frame that satisfies the above property. Let \( M \) be a model based on it and choose an arbitrary world \( \alpha \) in \( M \) and an arbitrary formula \( A \). Suppose that \( \models_M^{\alpha} [\Box (A \rightarrow [1] A \land ... \land [n] A)] \) and, for all \( i = 1, ..., n, \models_M^{\alpha} [A] \). We want to show that \( \models_M^{\alpha} [A] \). If there is no world which is \( R_* \)-accessible from \( \alpha \) then there is nothing to prove. Otherwise, let \( \beta \) be an arbitrary world such that \( \alpha R_* \beta \). We want to show that \( \models_M^{\beta} A \). By the assumed property, there exists a sequence \( \langle \delta_1, ..., \delta_m \rangle \)

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\(^5\) This property is a special case of a property considered by van der Hoek (1993, Definition 4.2 (c), p. 183). I am grateful to Joe Halpern for this reference.
in W and a sequence \( \langle i_1, \ldots, i_m \rangle \) in \( \{1, \ldots, n\} \) such that:
1. \( \delta_1 = \alpha \),
2. \( \delta_m = \beta \),
3. for every \( k = 2, \ldots, m \text{, } \alpha R_k \delta_k \), and
4. for every \( k = 1, \ldots, m-1 \text{, } \delta_k R_k \delta_{k+1} \).

Since \[ \models^M_\alpha \Box A \implies \models^M_\delta A \text{.} \]
and \[ \alpha R_k \models^M_\delta A \text{.} \]
Thus \[ \models^M_\delta i_j A \text{.} \]
and, therefore, \[ \models^M_\delta i_j A \text{.} \]
Hence \[ \models^M_\delta A \text{.} \]
Repeating this argument we obtain \[ \models^M_\delta A \text{, i.e. } \models^M_\beta A \text{, as desired.} \]

Now let \( \langle W, R_1, \ldots, R_n, R_\ast \rangle \) be a frame where \( R_\ast \) does not satisfy the above property.

Then there exist \( \alpha \) and \( \beta \) such that:
1. \( \alpha R_\ast \beta \),
2. either (b.1) there is no R-path from \( \alpha \) to \( \beta \) (recall that \( R = R_1 \cup \ldots \cup R_n \)) or
3. (b.2) if \( \langle \delta_1, \ldots, \delta_m \rangle \) is an R-path in W with \( \delta_1 = \alpha \) and \( \delta_m = \beta \) then, for some \( k = 2, \ldots, m \), not \( \alpha R_\ast \delta_k \). Let \( W_0 = \{ \gamma \in W \mid \alpha R_\ast \gamma \text{ and either there is no R-path from } \alpha \text{ to } \gamma \text{ or, if there is such a path, then not } \alpha R_\ast \delta \text{ for some } \delta \neq \alpha \text{ on this path} \} \). Thus \( \beta \in W_0 \).

Let \( p \) be a sentence letter and \( M \) a model based on this frame where \( F(p) = W - W_0 \). 

**STEP 1:** we show that \[ \models^M_\alpha \Box (p \rightarrow i_1 p \land \ldots \land i_n p) \] Choose an arbitrary \( \gamma \) such that \( \alpha R_\ast \gamma \). We have to prove that \[ \models^M_\gamma (p \rightarrow i_1 p \land \ldots \land i_n p) \] Suppose that \( \models^M_\gamma p \). Then \( \gamma \in W_0 \). It follows that there is an R-path \( \langle \delta_1, \ldots, \delta_m \rangle \) from \( \alpha (= \delta_1) \) to \( \gamma (= \delta_m) \) such that \( \alpha R_\ast \delta_k \) for all \( k = 2, \ldots, m \). Suppose that, for some

\[ i = 1, \ldots, n, \] \[ \models^M_\gamma i_1 p \text{.} \]

Then there exists an \( \eta \) such that \( \gamma R_\eta \) and \[ \models^M_\eta p \text{.} \]

Hence \( \alpha R_\ast \eta \). But then there is an R-path from \( \alpha \) to \( \eta \) with \( \alpha R_\ast \delta \) for every \( \delta \neq \alpha \) on this path, implying that \( \eta \in W_0 \), a contradiction. 

**STEP 2:** we show that, for every \( i = 1, \ldots, n, \) \[ \models^M_\alpha i_1 p \text{.} \]

Choose arbitrary \( \delta \in W \) and \( i \in \{1, \ldots, n\} \) such that \( \alpha R_i \delta \). We want to show that \[ \models^M_\delta p \text{.} \]

Hence \( \alpha R_\ast \delta \). But then, since there is an R-path from \( \alpha \) to \( \delta \) and \( \alpha R_\ast \delta \), it follows that \( \delta \in W_0 \), a contradiction. 

**STEP 3:** we show that \[ \models^M_\alpha \Box p \]. This follows from the fact that \( \beta \in W_0 \).
Thus the formula \( \star (p \rightarrow \bigwedge p \wedge ... \wedge n \ p) \rightarrow (\bigwedge p \wedge ... \wedge n \ p \rightarrow \Diamond p) \), which is an instance of axiom schema \( L \), is not true at \( \alpha \) in \( M \). ■

**REMARK 1.** Let \( R = R_1 \cup ... \cup R_n \) and \( R^{\text{Tr}} \) be the transitive closure of \( R \). That is, \( \alpha R^{\text{Tr}} \beta \) if and only if there exists a sequence \( \langle \delta_1, ... , \delta_m \rangle \) in \( W \) (with \( m \geq 2 \)) and a sequence \( \langle i_1, ... , i_{m-1} \rangle \) in \( \{1, ..., n\} \) such that: (1) \( \delta_1 = \alpha \), (2) \( \delta_m = \beta \) and (3) for every \( k = 1, ..., m-1 \), \( \delta_k R_{i_k} \delta_{k+1} \). Then it is easy to see\(^6\) that properties (i) and (ii) of Proposition 2 imply that \( R^{\text{Tr}} \subseteq R^* \), while Property (iii) implies that \( R^* \subseteq R^{\text{Tr}} \). Thus the conjunction of the three properties implies that \( R^* = R^{\text{Tr}} \) (clearly, \( R^{\text{Tr}} \) satisfies these three properties).

**PROPOSITION 3.** Let \( K_{n^*+S+P+L} \) be the system obtained by adding to \( K_{n^*} \) axiom schemata \( L, S_i \) and \( P_i \) for every \( i = 1, ..., n \). Then \( K_{n^*+S+P+L} \) is sound and complete with respect to the class of models where \( R^* \) is the transitive closure of \( R = R_1 \cup ... \cup R_n \).

**Proof.** (A) Completeness. Lismont (1993) proved (soundness and) completeness for the system obtained by adding to \( K_{n^*} \) the axiom schema

\[
F \quad \star A \rightarrow \square (A \wedge \star A)
\]

(where \( \square A \) is defined as \( \bigwedge \ A \wedge ... \wedge n \ A \) and the rule of inference

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\(^6\) *Proof.* Suppose that \( \alpha R^{\text{Tr}} \beta \), that is, there exists a sequence \( \langle \delta_1, ... , \delta_m \rangle \) in \( W \) (with \( m \geq 2 \)) and a sequence \( \langle i_1, ... , i_{m-1} \rangle \) in \( \{1, ..., n\} \) such that: (1) \( \delta_1 = \alpha \), (2) \( \delta_m = \beta \) and (3) for every \( k = 1, ..., m-1 \), \( \delta_k R_{i_k} \delta_{k+1} \). We want to show that \( \alpha R^* \beta \). Since \( \delta_{m-1} R_{i_{m-1}} \delta_m \), by (i) of Proposition 2, \( \delta_{m-1} R^* \delta_m \). By (ii), since \( \delta_{m-2} R_{i_{m-2}} \delta_{m-1} \) and \( \delta_{m-1} R^* \delta_m \), it follows that \( \delta_{m-2} R^* \delta_m \). Repeating this argument \( (m-1) \) times we obtain \( \delta_1 R^* \delta_m \), that is, \( \alpha R^* \beta \). Since \( \delta_1 = \alpha \) and \( \delta_m = \beta \).
\[
\begin{align*}
A & \to \Box A \\
\Box A & \to \Box A
\end{align*}
\]

Now, F is implied by the conjunction of \(S_i\) and \(P_i\) for all \(i = 1, ..., n\), while the above rule of inference is a derived rule in the system \(K_{n^n}\). Assume that \((A \to \bigwedge_i A \land ... \land \bigwedge_i A)\) is a theorem of \(K_{n^n}\). Then, by RNs, so is \(\bigwedge_i A \to \bigwedge_i A\). Hence, by L and MP, also \(\bigwedge_i A \land ... \land \bigwedge_i A \to \bigwedge_i A\) is a theorem.

(B) Soundness. This follows from Proposition 2 and Remark 1.\(^7\)

While Proposition 3 dealt with the system \(K_{n^n}\), the following proposition concerns the system \(K_{n^n}\).

**PROPOSITION 4.** For every sequence \(\langle i_1, i_2, ..., i_k \rangle\) of elements of \\{1,...,n\} and for every formula \(A\), the following is a theorem of \(K_{n^n}\) (which is the system obtained by adding to \(K_{n^n}\) the schemata \(S_i\) and \(P_i\) for all \(i = 1, ..., n\))

\[
\bigwedge_i A \to \bigwedge_i A
\]

**Proof.** If \(k=1\) this is axiom \(S_i\). We prove by induction that if the proposition is true for an arbitrary sequence \(\langle i_2, ..., i_k \rangle\) with \((k-1)\) elements (with \(k \geq 2\)) then it is true for the sequence \(\langle i_1, i_2, ..., i_k \rangle\) with \(k\) elements with arbitrary \(i_1\). Let \(\langle i_2, ..., i_k \rangle\) be an arbitrary

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\(^7\) Another way to prove soundness is to prove syntactically that \(L\) is a theorem of the system \(K_{n^n}\) plus the inference rule \(A \to \bigwedge_i A\). Such a (non-trivial) syntactical proof can be found in Lismont (1992).
sequence in \{1,\ldots,n\} and suppose that for every formula \(A\), \(\bigcirc A \rightarrow i_1 \ldots i_n A\) is a theorem of \(K_{\bigcirc}^n + S + P\). Let \(i_1\) be an element of \{1,\ldots,n\}. Then

1. \(\bigcirc A \rightarrow i_1 \ldots i_n A\) \hspace{1cm} \text{hypothesis}

2. \(i_1 \bigcirc A \rightarrow i_1 i_2 \ldots i_n A\) \hspace{1cm} 1, RM_{i_1} (see Chellas, 1980, p.114)

3. \(\bigcirc A \rightarrow i_1 \bigcirc A\) \hspace{1cm} \text{instance of } P_{i_1}

4. \(\bigcirc A \rightarrow i_1 i_2 \ldots i_n A\) \hspace{1cm} 2,3, PL. \(\blacksquare\)

REMARK 2. Recall that, at an informal level, \(\bigcirc A\) is thought of as the infinite conjunction of all formulae of the form \(i_1 i_2 \ldots i_n A\), for every possible sequence \(\langle i_1, i_2, \ldots, i_n \rangle\) in \{1,\ldots,n\} (that is, something is commonly believed if everybody believes it, everybody believes that everybody believes it, and so on ad infinitum). By Proposition 4, \(\bigcirc A\) implies this “infinite conjunction” (that is, each element of this infinite conjunction, which is not itself a formula) in the system \(K_{\bigcirc}^n + S + P\). Axiom L is not needed for this implication. In virtue of Proposition 3, adding axiom L has the effect of yielding the converse implication from the infinite conjunction to \(\bigcirc A\). To see that this converse implication does not hold in \(K_{\bigcirc}^n + S + P\) consider the following frame: \(n=2\), \(W = \{\alpha, \beta\}\), \(R_1 = R_2 = \emptyset\), \(R_\bigcirc = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}\). This frame satisfies properties (i) and (ii) of Proposition 2 (\(R_\bigcirc\) contains the transitive closure of \(R_1 \cup R_2\)) and therefore any model based on it validates all the theorems of \(K_{\bigcirc}^n + S + P\). Let \(p\) be a sentence letter and let \(M\) be a model where \(p\) is true at \(\alpha\) and false at \(\beta\). Then for every sequence \(\langle i_1, i_2, \ldots, i_n \rangle\) in \{1,2\}, the formula \([i_1 i_2 \ldots i_n] p\) is valid in \(M\). However, \(\bigcirc p\) is false at every world. In order for the implication from the infinite conjunction of all formulae of the form \(i_1 i_2 \ldots i_n A\) to \(\bigcirc A\) to hold, it is necessary that \(R_\bigcirc\) be contained in the
transitive closure of $R = R_1 \cup \ldots \cup R_n$ and this is precisely the role of axiom $L$ (cf. Remark 1).

It is easy to check, using Proposition 2, that the axiom schemata $S$, $P$ and $L$ form an independent set. For example, to see that $L$ is not a theorem of $K_{n^*} + S + P$, consider the following frame: $n = 1$, $W = \{\alpha, \beta\}$, $R_1 = \{(\alpha, \alpha)\}$ and $R_\alpha^* = \{(\alpha, \alpha), (\alpha, \beta)\}$ (cf. Figure 1).

\[ R_1: \quad \alpha \quad \beta \]

\[ R_\alpha^*: \quad \alpha \quad \beta \]

\[ p \quad \neg p \]

FIGURE 1

Note that $R_{\alpha}^*$ satisfies properties (i) and (ii) of Proposition 2, hence this frame validates $S_1$ and $P_1$. Thus every theorem of $K_{n^*} + S + P$ is valid in every model based on this frame. If $L$ were a theorem of $K_{n^*} + S + P$, then $L$ would have to be valid in every model based on this frame, which is not the case. In fact, let $p$ be a sentence letter and $M$ a model based on this frame where $F(p) = \{\alpha\}$. Then $\vdash_M^{\alpha} [1] \ p$, and therefore $\vdash^{\alpha}_M (p \rightarrow [1] \ p)$. Also, $\vdash^{\alpha}_\beta (p \rightarrow [1] \ p)$, since $\not\vdash^{\beta}_p$. Thus $\vdash^{\alpha}_\alpha [\square] \ p$. However, $\not\vdash^{\alpha}_\alpha [\boxdot] \ p$. Hence $\not\vdash^{\alpha}_\alpha ([1] \ p \rightarrow [\boxdot] \ p)$. It follows that $\vdash^{\alpha}_\alpha [\boxdot] (p \rightarrow [1] \ p)$, hence $\vdash^{\alpha}_\alpha ([1] \ p \rightarrow [\boxdot] \ p)$. It
References


