

SYNCHRONIC INFORMATION, KNOWLEDGE AND COMMON KNOWLEDGE IN EXTENSIVE GAMES

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Restricting attention to the class of extensive games defined by von Neumann and Morgenstern with the added assumption of perfect recall, we specify the information of each player at each node of the game-tree in a way which is coherent with the original information structure of the extensive form. We show that this approach provides a framework for a formal and rigorous treatment of questions of knowledge and common knowledge at every node of the tree. We construct a particular information partition for each player and show that it captures the notion of maximum information in the sense that it is the finest within the class of information partitions that satisfy four natural properties. Using this notion of “maximum information” we are able to provide an alternative characterization of the meet of the information partitions.

1. Introduction

The language of extensive games is complex and rich. It allows one to express such notions as the order of moves, the information a player has when it is her turn to move, etc. It is not, however, a sufficiently rich language in the sense that there are meaningful and natural statements that one can make (about a given extensive game) whose truth cannot be decided without making the language richer. We shall give two examples. Consider first the extensive form of Figure 1.

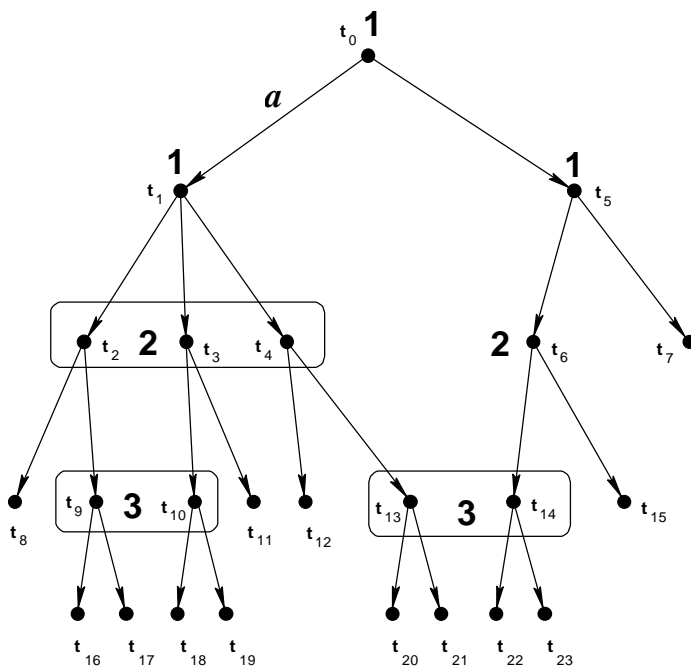


FIGURE 1

Consider the following statement:

« If node t_2 is reached, then every player knows that player 1 chose action a , but it is *not* common knowledge among the players that player 1 chose a ».

Is this statement true for the extensive game of Figure 1? If one adopts the standard semantics for knowledge and common knowledge¹ then, in order to answer this question, one needs the following:

1. a set of “states” (or “possible worlds”), Ω ,

2. to interpret the sentences “node t_2 is reached” and “player 1 chose a ” as events, that is, as subsets of Ω , call them τ and A , respectively,
3. for every player i , a partition \mathcal{K}_i of Ω (i 's information partition).

Once we have these three elements, deciding whether or not the above sentence is true becomes a simple matter of computation. In order for it to be true we need that:

- i. for every player i , $\tau \subseteq K_i A$ (where $K_i A$ is the event that player i knows A),
- ii. $\tau \not\subseteq CKA$ (where CKA is the event that A is common knowledge).

We can, trivially, construct a model where the above sentence is true and one where it is not:

VERIFYING model

REFUTING model

$$\Omega = \{ \alpha, \beta, \gamma \},$$

$$\text{“node } t_2 \text{ is reached”} = \tau = \{ \alpha \},$$

$$\text{“player 1 chose } a \text{”} = A = \{ \alpha, \beta \},$$

$$\mathcal{K}_1 = \mathcal{K}_2 = \{ \{ \alpha \}, \{ \beta, \gamma \} \},$$

$$\mathcal{K}_3 = \{ \{ \alpha, \beta \}, \{ \gamma \} \}.$$

$$\text{Meet of } \mathcal{K}_i \text{'s} = \{ \{ \alpha, \beta, \gamma \} \}.$$

$$\text{Thus } K_1 A = K_2 A = \{ \alpha \}, \quad K_3 A = \{ \alpha, \beta \},$$

$$CKA = \emptyset$$

$$\Omega = \{ \alpha \},$$

$$\text{“node } t_2 \text{ is reached”} = \tau = \{ \alpha \}$$

$$\text{“player 1 chose } a \text{”} = A = \emptyset$$

This is not, however, a satisfactory answer to the above question. Both of these models are artificial, in the sense that there is nothing in them that tells us that we are talking about the extensive form of Figure 1. There is no connection between the extensive form and the proposed model. On the other hand, it is quite natural to consider a model where Ω is the set of nodes of the game tree ($\Omega = \{t_1, t_2, \dots, t_{23}\} = T$) and those two sentences have the following interpretation:

*SENTENCE**INTERPRETING EVENT*

“ node t_2 is reached ”

$$\tau = \{ t_2 \}$$

“ player 1 chose a ”

$$A = \{ t_1, t_2, t_3, t_4, t_8, t_9, t_{10},$$

$$t_{11}, t_{12}, t_{13}, t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21} \}$$
 (set of successors of the root following arc a)

However, in order to analyze epistemic statements one would also need, for every player, a partition of the set of nodes T . The definition of extensive game does not provide us with such partitions: for every player, we are only given her information at her decision nodes and not at any other node.

As a further example of the fact that the language of extensive games is not sufficiently rich, consider the following statement:

« If a node of a subgame is reached, then every player knows
(or the stronger claim “it is common knowledge”) that the
subgame has been reached ».

Is this statement true in an arbitrary extensive game? Is it necessarily true? Again, it seems that a natural model in which to analyze such statement would be one where the set of states is the set of nodes. Once again we are lead to the issue of how to obtain, for every player, a partition of the set of nodes.

We provide a simple solution to this problem. We consider the class of extensive games defined by von Neumann and Morgenstern (1944) with the added assumption of perfect recall. Following Bonanno (1992a) we specify the information of each player at each node of the game-tree in a way which is coherent with the original information structure of the extensive form. Then the multi-stage structure of this class of games allows a characterization of the notions of knowledge and common knowledge for each stage of the game. However, the specification of this extended information structure is partially arbitrary. Therefore we consider a class of extended information structures that satisfy four natural properties and in Proposition 1 we provide a constructive characterization of the finest one (that is, the one which gives maximum information). It turns out that the same kind of construction also provides an

alternative characterization of the meet of these finest information partitions (and, therefore, of the notion of common knowledge with maximum information: Proposition 4). We also show that it is not necessarily true that at a node of a subgame it is common knowledge among the players that the subgame has been reached, although it is true for the case of maximum information. On the other hand, it is necessarily true (that is, for all extended information structures) that in a simultaneous game at every decision node there are no non-trivial events which are common knowledge among the players.

What can one expect to gain by extending the notion of information structure in extensive games? We suggest several answers.

1. Using extended information structures it is possible to trace the evolution of a player's beliefs along every possible play of the game. One can then impose (internal and mutual) consistency properties on belief revision and use these properties to define new equilibrium concepts (see Bonanno, 1992b) or provide alternative characterizations of known solution concepts (see Bonanno, 1995).

2. In many cases it may be reasonable to assume that the extensive game is not an exhaustive description of the strategic situation. For example, unmodelled phases of communication and/or bargaining may occur. This is the rationale for the definition of coalition-proof and renegotiation-proof equilibria². Under this interpretation a synchronic description of players' information is essential and constitutes a pre-requisite for any extension of the above-mentioned solution concepts to situations of dynamic strategic interaction with differential information. For example, one might impose the restriction that a coalition of players can contemplate, at node t , a joint deviation from a given equilibrium only if it is common knowledge among those players, at node t , that the joint deviation will benefit them all. Such an approach would clearly require extended information structures, without which the notion of an event being common knowledge at an arbitrary node is not well-defined.

3. The literature on conjectural or self-confirming equilibria [Battigalli and Guaitoli (1996), Fudenberg and Levine (1993), Kalai and Lehrer (1993a,b), Rubinstein and Wolinsky (1994)] interprets the given extensive game as a constituent of a larger repeated game (with, possibly, imperfect monitoring) and tries to define and characterize outcomes that can be "stable"

under plausible learning processes. In this literature information partitions on the set of terminal nodes of the game are taken as primitive and issues of (common) knowledge are relevant. The approach put forward in this paper extends to the entire game – and provides a rationale for – the information partitions on the set of end nodes of the constituent game.

4. The notion of extended information structure can considerably simplify the representation of games. Suppose, for instance, that we want to describe a parlor game as an extensive game where at the beginning a die is thrown, every player observes the result and then, say, player 1 has to make the next move. That every player observes the outcome of the throw of the die can be modeled in a standard extensive game by drawing after each of the six moves of Nature a sequence of information sets for each player, where each information set consists of a single node and is followed by a single action. Each of these information sets indicates that the corresponding player gets informed, not that he has to make a decision. After that we model the remaining game, starting with the information sets for player 1. Of course, hardly anybody will do so in practice. One will not draw any information sets followed by single actions and will draw information sets for the opponents of player 1 only after player 1 has moved. But this practice does not properly reflect the fact that the information about the throw of a die and player 1's move do not come together, but in pieces. One might argue that the only information about the play of the game which can be relevant is the one that a player has when he has to make a "real" choice. And this information is described by the information sets followed by at least two actions. However, there is at least one interesting solution concept which is not invariant under the addition or deletion of information sets followed by single actions. This is the notion of perfect sequential equilibrium due to Grossman and Perry (1986; on this point see Noeldeke and van Damme, 1990). Instead of drawing information sets followed by single actions we can use the extended partitions. Then it suffices to draw, in our example, one decision node (the root) for the random move followed by six decision nodes of player 1 which are followed by his actions and corresponding decision nodes. Then the description of the remaining game follows. That, for instance, everybody is informed about the outcome of the random move is described by having the six decision nodes following the origin as singleton sets in the extended partitions. Besides saving nodes and information sets the extended partitions provide directly, for each decision node, the information each player has about the play of the game until this node is reached. The extended partitions may constitute a step towards a general, alternative way to model

games in extensive form, where one is not forced to misrepresent the actual flow and timing of information.

5. Recently, Maskin and Tirole (1994) have analyzed the notion of sequential Markov equilibrium in multistage games with simultaneous choices at every stage. Crucial to their analysis is the notion that at every stage each player has an information partition of the histories up to that point. Our approach clearly generalizes the notion of information partition over the set of histories to a larger class of games and provides a test for checking whether any given postulated partition is consistent with the original information structure of the extensive game (in order to be, it has to be a coarsening of the maximum information extended structure that we define and characterize in Section 4).

6. In a recent paper, Aumann (1995) models explicitly the notion of common knowledge of rationality in extensive games and studies its implications. Aumann's epistemic model refers to the *ex ante* stage (before the game is played) and is applied only to perfect information games. He suggests (p. 17) that his results would still hold if his epistemic model were to be extended so as to encompass the play of the game, since "when the time comes for a player to move, he certainly knows at least as much as he did when play started". The approach we put forward enables one to extend any epistemic model of a game to include the set of nodes of the game tree³. Furthermore, it allows one to remove the restriction to perfect information games (for an example see Bonanno, 1994).

7. Finally, as shown above, without extended information structures there are meaningful statements about extensive games whose truth cannot be decided.

2. Definition of von Neumann-Morgenstern extensive form with perfect recall

For reasons that will be discussed later (Section 6), throughout this paper we restrict attention to the class of extensive forms⁴ (with or without chance moves) defined by von Neumann and Morgenstern (1944) with the added assumption of perfect recall. We shall call them "vN-M extensive forms".

DEFINITION. An extensive form is a *vN-M extensive form* if, whenever two nodes belong to the same information set, the number of predecessors of

one is equal to the number of predecessors of the other. Formally: $\forall t, t' \in T$, $H(t) = H(t') \Rightarrow \ell(t) = \ell(t')$ (where T is the set of nodes of the game tree and, for every $t \in T$, $H(t)$ is the information set that contains t and $\ell(t)$ is the number of predecessors of t)⁵.

REMARK 1. Every extensive form with perfect information is a vN-M extensive form.

REMARK 2. Given an extensive form which is *not* a vN-M extensive form, it may be possible to transform it into one by means of an “inessential transformation”. Consider, for example, the extensive form of Figure 2a. It is not a vN-M extensive form because $\ell(t_2) = 1$ and $\ell(t_3) = 2$ and t_2 and t_3 belong to the same information set (of player 2). However, it can be transformed into one by adding a “dummy” node and assigning to it a “dummy” player with only one choice, as shown in Figure 2b (the dummy node is t_4 and the dummy player is 3). The interchange of simultaneous moves (see Thompson, 1952, and Bonanno, 1992c) is another “inessential transformation” that sometimes can be used to transform an extensive form which is not a vN-M extensive form into one which is, as shown in Figure 3 (the extensive form of Figure 3a is not, while that of Figure 3b is, a vN-M extensive form). On the other hand, Figure 4 shows an extensive form which is not a vN-M extensive form and cannot be transformed into one by means of one of the transformations mentioned above.

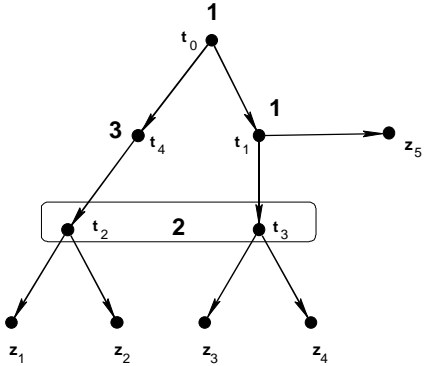
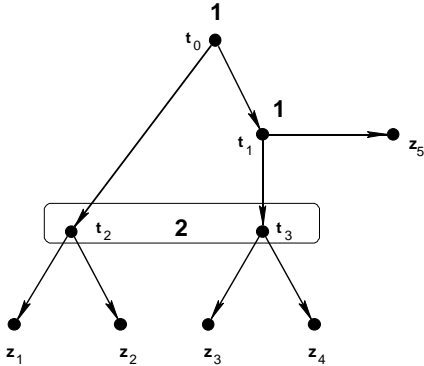


FIGURE 2a

FIGURE 2b

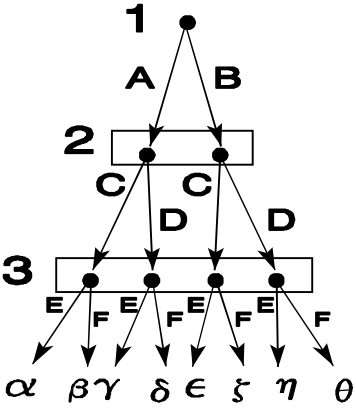
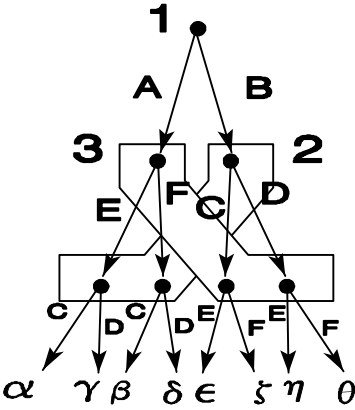


FIGURE 3a

FIGURE 3b

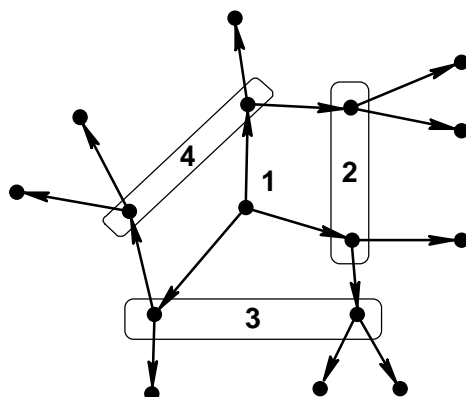


FIGURE 4

From now on by “extensive form” we shall always mean a vN - M extensive form with perfect recall.

3. Information partitions

As explained in the Introduction, our objective is to complete the definition of extensive game by adding, for every player i , a partition of the set T of nodes (player i 's information partition). Clearly, the information structure of the game and the assumption of perfect recall impose some constraints.

DEFINITION. Let G be a vN - M extensive form with perfect recall. An *information completion* of G is an n -tuple $\langle \mathcal{K}_1, \dots, \mathcal{K}_n \rangle$, where, for each player $i = 1, \dots, n$, \mathcal{K}_i is a partition of the set of nodes T , satisfying the following properties (for every node $t \in T$, we denote by $[t]_i$ the cell of \mathcal{K}_i that contains t):

(1) *Coherence with the information structure*⁶: if node t belongs to information set h of player i , then the cell of \mathcal{K}_i that contains t coincides with h . Formally (see the Appendix for a list and explanation of the notation):

$$t \in h \in \mathbb{H}_i \Rightarrow [t]_i = h.$$

(2) *Players remember what choices they made*: if the immediate predecessor of t belongs to information set h of player i and t comes after choice c at h , then every $x \in [t]_i$ comes after choice c . Formally:

$$(p(t) \in h) \wedge (t \in S(h,c)) \Rightarrow [t]_i \subseteq S(h,c).$$

[where $p(t)$ denotes the immediate predecessor of t and $S(h,c)$ the set of immediate successors of nodes in h following choice c at h].

(3) *Players do not forget*: if node x is a successor of node t , then every terminal node that can be reached from a node in $[x]_i$ must also be reachable from some node in $[t]_i$ (equivalently, every node in $[x]_i$ must have a predecessor in $[t]_i$). Formally:

$$t \prec x \Rightarrow (\forall x' \in [x]_i, \exists t' \in [t]_i : t' \prec x')$$

[where \prec denotes the precedence relation].

(4) *Players know the stage of the game*: for every node t , the cell containing t is a subset of the set of nodes that belong to the same stage as t . Formally:

$$\ell(t) = k \Rightarrow [t]_i \subseteq T^k$$

[recall that $\ell(t)$ denotes the number of predecessors of t , that is, the stage to which t belongs; T^k is the set of stage- k nodes: $T^k = \{t \in T \mid \ell(t) = k\}$].

By (1), $\mathcal{K}_i \supseteq \mathbb{H}_i$ (where \mathbb{H}_i is the set of player i 's information sets). Thus the elements of \mathcal{K}_i represent a generalization of the notion of information set. Properties (2) and (3) are natural restrictions in view of the fact that we only consider extensive forms with perfect recall. Property (4) is a reflection of the

multi-stage structure of the game (it is an immediate consequence of the definition of vN-M extensive game that when player i has to move she knows the stage reached by the play; condition (4) extends this property to nodes that are not i 's decision nodes).

Given an extensive form G in general there are several possible information completions of it. Consider, for example, the extensive form of Figure 5. To simplify, restrict attention to the set D of decision nodes (the nodes labeled t_0 to t_6). It is easy to see that properties (1)-(4) above imply that, for any information completion $\langle \mathcal{K}_1, \mathcal{K}_2 \rangle$ of this extensive form, \mathcal{K}_1 is the finest partition, that is, for every $t \in D$, $[t]_1 = \{t\}$. On the other hand, for player 2 there are several possibilities. We list two below:

$$\begin{aligned} \overline{\mathcal{K}}_2 &= \{\{t_0\}, \{t_1, t_2\}, \{t_3\}, \{t_4\}, \{t_5\}, \{t_6\}\} && \text{MAXIMUM information} \\ \underline{\mathcal{K}}_2 &= \{\{t_0\}, \{t_1, t_2\}, \{t_3, t_5\}, \{t_4, t_6\}\} && \text{MINIMUM information} \end{aligned}$$

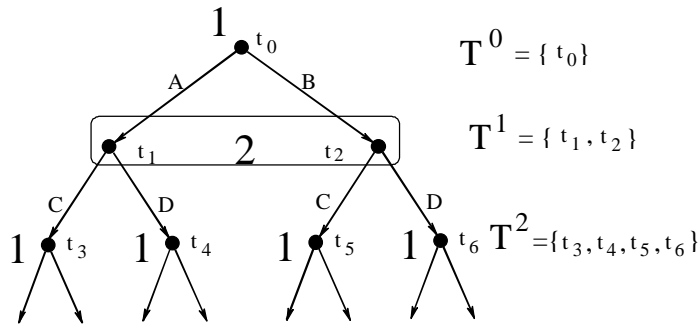


FIGURE 5

If the information transmission rule embodied in $\overline{\mathcal{K}}_2$ is adopted, then more information is conveyed to player 2 than in the case where the rule expressed by $\underline{\mathcal{K}}_2$ is followed. In other words, $\overline{\mathcal{K}}_2$ is a refinement of $\underline{\mathcal{K}}_2$ (or the latter is a

coarsening of the former). We shall show in the next section how to generate the maximum information completion for any given extensive form.

Given an extensive form G and an information completion $\langle \mathcal{K}_1, \dots, \mathcal{K}_n \rangle$ of G , for every player i we can define a *knowledge operator* $K_i: \wp(T) \rightarrow \wp(T)$ [where $\wp(T)$ denotes the set of subsets of T] as follows:

$$K_i E = \{ t \in T \mid [t]_i \subseteq E \}.$$

Thus $K_i E$ is the event that player i knows E . Let \mathcal{M} be the *meet* (that is, the finest common coarsening) of the partitions $\mathcal{K}_1, \dots, \mathcal{K}_n$. For every node t , denote by $[t]_{\mathcal{M}}$ the cell of \mathcal{M} containing t . Following Aumann (1976), we define a *common knowledge operator* $CK: \wp(T) \rightarrow \wp(T)$ as follows:

$$CKE = \{ t \in T \mid [t]_{\mathcal{M}} \subseteq E \}.$$

Thus CKE is the event that it is common knowledge among all the players that event E has occurred⁷.

The extensive form of Figure 5 gives us an opportunity to answer one question raised in the Introduction, namely whether it is necessarily true that if the play of the game reaches a node that belongs to subgame G' , then every player knows (or the stronger claim “it is common knowledge”) that G' has been reached. The answer is “No”. To see this, let G' be the subgame with root t_3 . Interpret the sentence “node t_3 is reached” as the event $\tau = \{t_3\}$ and the sentence “subgame G' has been reached” as the event Γ consisting of t_3 and its successors. Choose the information completion where player 2’s partition is given by $\underline{\mathcal{K}}_2$ (given above). Then $t_3 \in K_1 \Gamma$, but $t_3 \notin K_2 \Gamma$. Thus at node t_3 not every player knows (*a fortiori* it is not common knowledge) that G' has been reached. However, we will see later that in the case of maximum information it is indeed true, in every extensive game, that at a node of a subgame it is common knowledge among the players that the subgame has been reached.

4. Maximum information

In this section we construct, for every vN-M extensive form with perfect recall, a particular information completion of it and show that it is the finest (that is, the most informative) of the information completions that satisfy Properties (1)-(4).

First some notation. For every node t and for every player i , let $\mathcal{H}_i(t)$ be the set of information sets of player i that are crossed by paths starting at t ($t \lesssim y$ means either $t = y$ or $t \prec y$):

$$\mathcal{H}_i(t) = \{ h \in \mathbb{H}_i \mid \exists y \in h : t \lesssim y \}.$$

Next we introduce, for every player i , a relation on T , denoted by \sim_i . Let $v, w \in T$. Then $v \sim_i w$ if and only if, either (i) $v = w$, or (ii) $\ell(v) = \ell(w)$ and $\mathcal{H}_i(v) \cap \mathcal{H}_i(w) \neq \emptyset$. The relation \sim_i is clearly reflexive and symmetric. However, in general, it is not transitive. For example, in Figure 6 we have that $y_1 \sim_2 y_2$ and $y_2 \sim_2 y_3$ but **not** $y_1 \sim_2 y_3$.

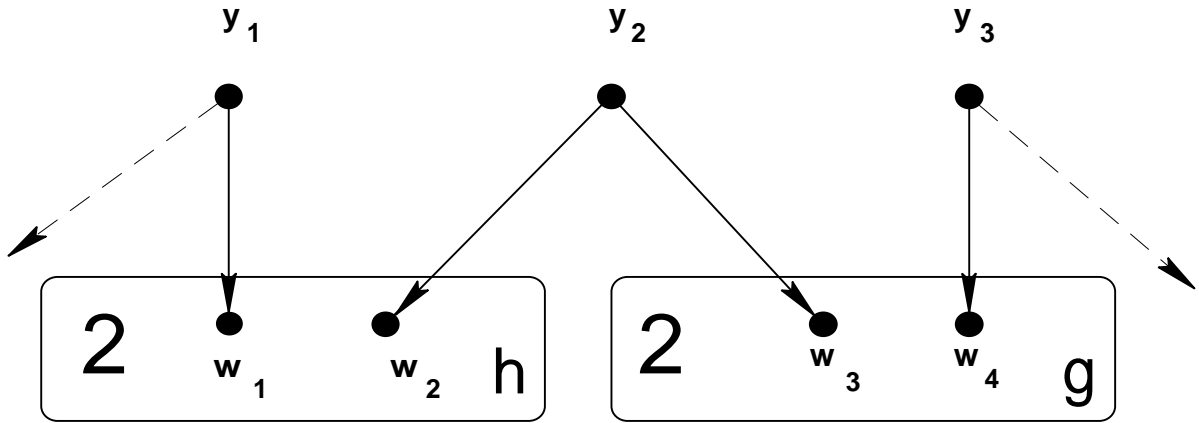


FIGURE 6

Let \sim_i^* denote the transitive closure of \sim_i . Thus $v \sim_i^* w$ if and only if there exists a finite sequence of nodes $\langle y_1, y_2, \dots, y_m \rangle$ such that $y_1 = v$, $y_m = w$ and, for all $k = 1, \dots, m-1$, $y_k \sim_i y_{k+1}$. Then \sim_i^* is an equivalence relation on T . Let $H_i(t)$ denote the equivalence class of t generated by \sim_i^* and $\overline{\mathcal{K}}_i$ the set of equivalence classes, that is,

$$H_i(t) = \{ v \in T^{\ell(t)} \mid v \sim_i^* t \}, \text{ and}$$

$$\overline{\mathcal{K}}_i = \{ A \subseteq T \mid A = H_i(t) \text{ for some } t \in T \}.$$

In Lemma 1 and Proposition 1 below we show that $\langle \overline{\mathcal{K}}_1, \dots, \overline{\mathcal{K}}_n \rangle$ is an information completion [that is, it satisfies properties (1)-(4)] and that it is the finest of all the information completions that satisfy those properties. First we give an illustration based on Figures 1 and 2b.

EXAMPLE 1. Consider the game of Figure 2b, where $T^0 = \{t_0\}$, $T^1 = \{t_1, t_4\}$, $T^2 = \{t_2, t_3, z_5\}$ and $T^3 = \{z_1, z_2, z_3, z_4\}$. Let $\overline{\mathcal{K}}_i^k$ denote the subset of $\overline{\mathcal{K}}_i$ that gives a partition of T^k . Then, for $i = 2$ we have:

$$\begin{aligned} \overline{\mathcal{K}}_2^0 &= \{H_2(t_0) = T^0\} \\ \overline{\mathcal{K}}_2^1 &= \{H_2(t_1) = H_2(t_4) = T^1\} \\ \overline{\mathcal{K}}_2^2 &= \{H_2(t_2) = H_2(t_3) = \{t_2, t_3\}, H_2(z_5) = \{z_5\}\} \\ \overline{\mathcal{K}}_2^3 &= \{H_2(z_1) = \{z_1\}, H_2(z_2) = \{z_2\}, H_2(z_3) = \{z_3\}, H_2(z_4) = \{z_4\}\}. \end{aligned}$$

EXAMPLE 2. Consider the game of Figure 1 and stage 2. Then:

$$\begin{aligned} \overline{\mathcal{K}}_1^2 &= \{\{t_2\}, \{t_3\}, \{t_4\}, \{t_6\}, \{t_7\}\}, \quad \overline{\mathcal{K}}_2^2 = \{\{t_2, t_3, t_4\}, \{t_6\}, \{t_7\}\} \\ \overline{\mathcal{K}}_3^2 &= \{\{t_2, t_3\}, \{t_4, t_6\}, \{t_7\}\}, \quad \text{thus the meet is } \overline{\mathcal{M}}^2 = \{\{t_2, t_3, t_4, t_6\}, \{t_7\}\} \end{aligned}$$

Using $\overline{\mathcal{K}}_i$ we can show that the statement (which was discussed in the Introduction) “if node t_2 is reached then every player knows that player 1 chose

action a but it is not common knowledge that player 1 chose a ” is indeed true. Interpret the sentence “node t_2 is reached” as the event $\{t_2\}$ and the sentence “player 1 chose action a ” as the event $A = \{t_1, t_2, t_3, t_4, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21}\}$, that is, A consists of the successors of the root following the arc that represents choice a . Then we have that $K_1 A \cap T^2 = \{t_2, t_3, t_4\} = K_2 A \cap T^2$ and $K_3 A \cap T^2 = \{t_2, t_3\}$. Since t_2 belongs to all three sets, it is indeed true that all the players know, at t_2 , that player 1 chose a . On the other hand, $CKA \cap T^2 = \emptyset$ and, therefore, also the claim that at t_2 it is not common knowledge that player 1 chose a is true. This is because at t_2 player 2 considers t_4 possible and if t_4 is indeed the true state then player 3 considers t_6 possible and at t_6 it is not true that player 1 chose action a . Thus at t_2 player 2 does not know that player 3 knows that player 1 chose a .

REMARK 1. If t is a terminal node or the root of a subgame, then for every player i , $H_i(t) = \{t\}$.

LEMMA 1. The information partition $\overline{\mathcal{K}}_i = \{A \subseteq T \mid A = H_i(t) \text{ for some } t \in T\}$ constructed above satisfies properties (1)-(4) (cf. Section 3).

Proof. **(1)** Coherence with the information structure. Let $i \in N$ and $t \in h \in \mathbb{H}_i$. We want to show that $H_i(t) = h$. Let $k = \ell(t)$. By definition of vN-M extensive form, $h \subseteq T^k$. By definition of \sim_i , $y \sim_i x$ for every $y, x \in h$. Thus $H_i(t) \supseteq h$. Suppose that $H_i(t) \neq h$. Then there exist a $\tau \in H_i(t) \setminus h$, an $x \in h$ and a $g \in \mathbb{H}_i$ with $g \neq h$, such that both τ and x have a successor in g . Then the successors of x in g come after a choice at h , while the successors of τ in g do not, contradicting the hypothesis of perfect recall.

(2) Players remember what choices they made. Let i be a player, t a node, h an information set of player i and c a choice at h . We have to show that if $p(t) \in h$ and $t \in S(h, c)$ (recall that $p(t)$ denotes the immediate predecessor of t and $S(h, c)$ the set of immediate successors of nodes in h following choice c) then $H_i(t) \subseteq S(h, c)$. When t is a decision node of player i , this is a trivial consequence of (1) and perfect recall. Assume, therefore, that t is not a decision

node of player i . Fix an arbitrary $\tau \in H_i(t)$. Then there exists a sequence $\langle y_1, \dots, y_m \rangle$ with $y_1 = t$, $y_m = \tau$ and $y_k \sim_i y_{k+1}$ for all $k = 1, \dots, m-1$. We prove that $\tau \in S(h, c)$ by induction on m . The statement is trivially true for $m = 1$. Assume it is true for $m = r-1$. Let $\langle y_1, \dots, y_{r-1}, y_r \rangle$ be a chain such that $y_k \sim_i y_{k+1}$ for all $k = 1, \dots, r-1$. Since y_{r-1} is connected with t by a chain of length $r-1$, the induction hypothesis yields $y_{r-1} \in S(h, c)$. Since $y_{r-1} \sim_i \tau$, there is a $g \in \mathbb{H}_i$ which is intersected by a path from y_{r-1} and a path from τ . Since $y_{r-1} \in S(h, c)$, by perfect recall g contains only successors of nodes in $S(h, c)$, therefore also $\tau \in S(h, c)$.

(3) Players do not forget. Let $i \in N$ and $v, w \in T$ with $v \prec w$. We want to show that for every $w' \in H_i(w)$ there exists a $v' \in H_i(v)$ such that $v' \prec w'$. Fix an arbitrary $w' \in H_i(w)$. Let v' be the unique node such that $v' \prec w'$ and $\ell(v') = \ell(v)$. Let $\langle w_1, \dots, w_m \rangle$ be the sequence in $T^{\ell(w)}$ such that $w_1 = w'$, $w_m = w$ and, for all $j = 1, \dots, m-1$, $w_j \sim_i w_{j+1}$. Let $\langle v_1, \dots, v_m \rangle$ be the corresponding sequence in $T^{\ell(v)}$ with $v_j \prec w_j$ for all $j = 1, \dots, m$ (it is possible that $v_j = v_{j+1}$ for some j ; furthermore, $v_1 = v'$ and $v_m = v$). Note that, for every j , $\mathcal{H}_i(w_j) \subseteq \mathcal{H}_i(v_j)$ and $\mathcal{H}_i(w_j) \cap \mathcal{H}_i(w_{j+1}) \neq \emptyset$. Hence, for every j , $\mathcal{H}_i(v_j) \cap \mathcal{H}_i(v_{j+1}) \neq \emptyset$, that is, $v_j \sim_i v_{j+1}$. Therefore $v' \in H_i(v)$.

(4) Players know the stage of the game. This is true by construction. ■

The next proposition shows that the information partition $\overline{\mathcal{K}}_i$ captures the notion of maximum information that can be conveyed to the players.

PROPOSITION 1. Fix a vN - M extensive form with perfect recall and a player i . Let \mathcal{K}_i be an arbitrary partition of T that satisfies properties (1)-(3) [*a fortiori*, properties (1)-(4)]. Then \mathcal{K}_i is a coarsening of $\overline{\mathcal{K}}_i$, that is, for all $t \in T$, $[t]_i \supseteq H_i(t)$ (recall that $[t]_i$ denotes the cell of \mathcal{K}_i that contains t , while $H_i(t)$ denotes the cell of $\overline{\mathcal{K}}_i$ that contains t).

Proof. Fix an arbitrary $t \in T$. If $H_i(t) = \{t\}$, there is nothing to prove. If $t \in h \in \mathbb{H}_i$, then by property (1), $[t]_i = h$ and, by Lemma 1, $H_i(t) = h$. Thus it only

remains to consider the case where t is *not* a decision node of player i and $H_i(t) \neq \{t\}$. Fix an arbitrary $\tau \in H_i(t)$ with $\tau \neq t$. We want to show that $\tau \in [t]_i$. Let $k = \ell(t)$. By property (4), $[t]_i \subseteq T^k$. By definition of $H_i(t)$, there exists a finite sequence y_1, y_2, \dots, y_m in T^k such that: $y_1 = \tau$, $y_m = t$, and, for every $j = 2, \dots, m$, there exists an information set $h_j \in \mathbb{H}_i$ which is crossed by a path from y_{j-1} to a terminal node and by a path from y_j to a terminal node (see Figure 7, where $m = 4$). Let w_1 be a successor of y_1 in h_2 , w_m be a successor of y_m in h_m . For $j = 2, \dots, m-1$, let w_{jj} be a successor of y_j in h_j and w_{jj+1} be a successor of y_{j+1} in h_{j+1} (see Figure 7). By property (1) (coherence with the information structure), $[w_1]_i = h_2$, $[w_m]_i = h_m$, and, for every $j = 2, \dots, m-1$, $[w_{jj}]_i = h_j$ and $[w_{jj+1}]_i = h_{j+1}$. By property (3) (players do not forget) every node in h_2 has a predecessor in $[y_1]_i$ and a predecessor in $[y_2]_i$. Thus $[y_1]_i \cap [y_2]_i \neq \emptyset$. Hence $[y_1]_i = [y_2]_i$. Similarly, by property (3), every node in h_3 has a predecessor in $[y_2]_i$ and a predecessor in $[y_3]_i$. Thus $[y_2]_i \cap [y_3]_i \neq \emptyset$. Hence $[y_2]_i = [y_3]_i$. Repeating the same argument for $j = 3, \dots, m$, we reach the conclusion that $[y_1]_i = [y_m]_i$, that is, $[\tau]_i = [t]_i$. ■

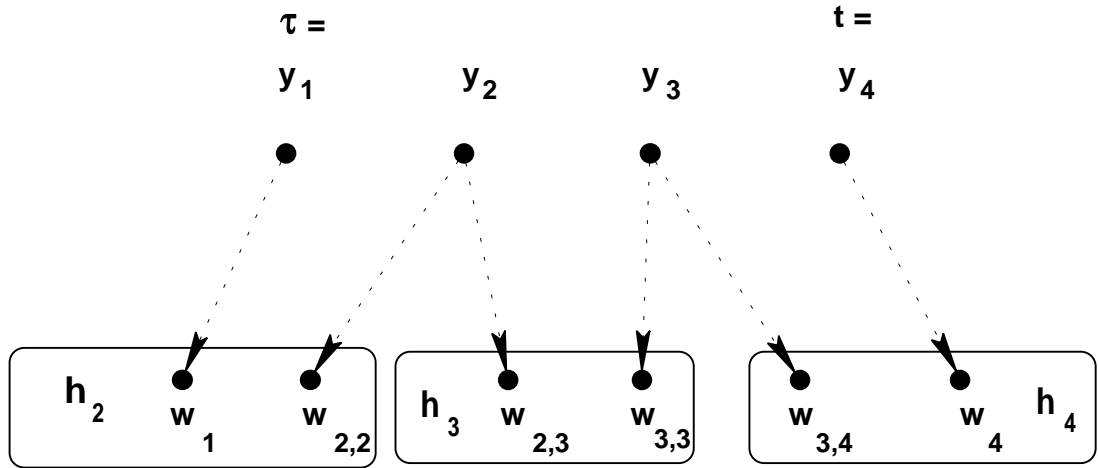


FIGURE 7

From now on, in virtue of Proposition 1, we shall call $\langle \overline{\mathcal{K}}_1, \dots, \overline{\mathcal{K}}_n \rangle$ the *maximum information completion* and $\overline{\mathcal{K}}_i$ the maximum information partition for player i .

We now highlight some further properties of the maximum information completion.

PROPOSITION 2. Fix a vN - M extensive form with perfect recall G and let $\langle \overline{\mathcal{K}}_1, \dots, \overline{\mathcal{K}}_n \rangle$ be the maximum information completion of G . Then, for every node \hat{t} , \hat{t} is the root of a sub**GAME** if and only if, at every node t such that $\hat{t} \lesssim t$, it is common knowledge among all the players that the play of the game has reached the sub**TREE** with root \hat{t} (that is, \hat{t} is the root of a subgame if and only if, for every t such that $\hat{t} \lesssim t$, $[t]_{\overline{\mathcal{M}}} \subseteq \{v \in T^{\ell(t)} \mid \hat{t} \lesssim v\}$).

Proof. (Sufficiency). Let \hat{t} be the root of a subgame. As remarked previously, it is an immediate consequence of the definition of $H_i(\cdot)$ that, for every $i \in N$, $H_i(\hat{t}) = \{\hat{t}\}$. Hence if $\overline{\mathcal{M}}$ is the meet (the finest common coarsening) of $\{\overline{\mathcal{K}}_i\}_{i \in N}$, then $\{\hat{t}\} \in \overline{\mathcal{M}}$. Furthermore, if t is a successor of \hat{t} , then, by property (3) (players do not forget), for every player i , every node in $H_i(t)$ is a successor of \hat{t} . Thus $[t]_{\overline{\mathcal{M}}}$ contains only successors of \hat{t} .

(Necessity). Suppose that \hat{t} is not the root of a subgame. Then there exist a player i , an $h \in \mathbb{H}_i$ and two nodes $w, w' \in h$ such that $\hat{t} \lesssim w$ and $\hat{t} \not\lesssim w'$. By

Proposition 1, $w' \in H_i(w) = h$. It follows that $w' \in [w]_{\overline{\mathcal{M}}}$. Therefore $[w]_{\overline{\mathcal{M}}} \not\subseteq \{v \in T^{\ell(w)} \mid \hat{t} \preceq w\}$. ■

We showed in the preceding section (cf. Figure 5) that Proposition 2 is not true for a general information completion. Proposition 3 below, on the other hand, is true for *all* information completions.

An extensive form is said to be *simultaneous* if every play crosses all the information sets (thus, by perfect recall, if there are at least two choices at each information set, it must be the case that every player has exactly one information set). Let $\langle \mathcal{K}_1, \dots, \mathcal{K}_n \rangle$ be an arbitrary information completion. Then for every decision node t , if $\ell(t) = k$, there must be a player i and an information set h of player i such that $h = T^k$. It follows that if \mathcal{M} is the meet of $\{\mathcal{K}_i\}_{i \in N}$, then $T^k \in \mathcal{M}$. Thus we have proved the following proposition.

PROPOSITION 3. Fix a simultaneous game G , an arbitrary information completion $\langle \mathcal{K}_1, \dots, \mathcal{K}_n \rangle$ of G and a decision node t . Let $k = \ell(t)$. Then the only stage- k event which is common knowledge among all the players is the certain event, that is, T^k . In other words, there are no non-trivial stage- k events that are common knowledge among all the players at stage k .

Finally it is worth repeating that for the maximum information partition $\overline{\mathcal{K}}_i$ the following is true: if z is a terminal node, then for every player i , $H_i(z) = \{z\}$. Thus, when the play of the game ends at z , it is common knowledge among all the players that the outcome is z ⁸.

5. Maximum information and common knowledge

In this section we show that if we restrict attention to the maximum information completion $\langle \mathcal{K}_1, \dots, \mathcal{K}_n \rangle$ constructed in the previous

section, then we can provide an alternative characterization of the meet $\overline{\mathcal{M}}$ of the partitions $\mathcal{K}_1, \dots, \mathcal{K}_n$ (and, to this extent, of the notion of common knowledge). Recall that, given an information completion $\langle \mathcal{K}_1, \dots, \mathcal{K}_n \rangle$ (not necessarily the maximum one) for every stage k and every player i , there is a subset of \mathcal{K}_i , call it \mathcal{K}_i^k , which is a partition of T^k . For every subset of players $J \subseteq N$, denote by \mathcal{M}_J the meet (finest common coarsening) of $\{\mathcal{K}_i\}_{i \in J}$. Then for every stage k there is a subset of \mathcal{M}_J , call it \mathcal{M}_J^k , which is a partition of T^k . If the information partition is the maximum one, namely $\langle \overline{\mathcal{K}}_1, \dots, \overline{\mathcal{K}}_n \rangle$, then we shall denote the corresponding meets by $\overline{\mathcal{M}}_J$ and $\overline{\mathcal{M}}_J^k$, respectively. We want to provide an alternative characterization of $\overline{\mathcal{M}}_J^k$ that is, a characterization which is *not* in terms of the concept of “common coarsening” of the partitions $\{\overline{\mathcal{K}}_i^k\}_{i \in J}$. In order to do this we repeat the construction of Section 4 with respect to an arbitrary set of player $J \subseteq N$ (instead of a single player i).

For every node t and for every set of players $J \subseteq N$, let $\mathcal{H}_J(t)$ be the set of information sets of players in J that are crossed by paths starting at t :

$$\mathcal{H}_J(t) = \{h \in \bigcup_{i \in J} \mathbb{H}_i \mid \exists y \in h : t \preceq y\}.$$

Next we introduce, for every $J \subseteq N$, a relation on T , denoted by \sim_J . Let $v, w \in T$ and $J \subseteq N$. Then $v \sim_J w$ if and only if, either (i) $v = w$, or (ii) $\ell(v) = \ell(w)$ and $\mathcal{H}_J(v) \cap \mathcal{H}_J(w) \neq \emptyset$. The relation \sim_J is clearly reflexive and symmetric. However, in general, it is not transitive. Let \sim_J^* denote the transitive closure of \sim_J . Thus $v \sim_J^* w$ if and only if there exists a finite sequence of nodes $\langle y_1, y_2, \dots, y_m \rangle$ such that $y_1 = v$, $y_m = w$ and, for all $k = 1, \dots, m-1$, $y_k \sim_J y_{k+1}$. Then \sim_J^* is an equivalence relation on T^k .

Let

$$\mathbf{H}_J(t) = \{ v \in T^{\ell(t)} \mid v \sim_J^* t \}$$

and

$$\mathbb{H}_J^k = \{ \mathbf{H}_J(t) \mid t \in T^k \}.$$

Proposition 4 shows that the equivalence relation \sim_J^* represents, at each stage, the smallest event which is common knowledge among the players in J , under maximum information.

PROPOSITION 4. For every stage k and every subset of players $J \subseteq N$, $\overline{\mathcal{M}}_J^k = \mathbb{H}_J^k$.

Proof. In order to simplify the notation, we shall denote the cell of $\overline{\mathcal{M}}_J^k$ containing node t as $\overline{\mathbf{M}}_J(t)$. It is well-known that $v \in \overline{\mathbf{M}}_J(t)$ if and only if there exists a finite sequence of nodes $\langle y_1, y_2, \dots, y_m \rangle$ and a finite sequence of players $\langle i_2, \dots, i_m \rangle$ in J , such that: $y_1 = v$, $y_m = t$, and, for every $j = 1, \dots, m-1$, $y_j \in \mathbf{H}_{i_{j+1}}(y_{j+1})$ [see Figure 8, where $i_2 = i_4 = 2$, $i_3 = i_5 = 1$, $v = y_1 \in \mathbf{H}_{i_2}(y_2) = \mathbf{H}_2(y_2)$, $y_2 \in \mathbf{H}_{i_3}(y_3) = \mathbf{H}_1(y_3)$, $y_3 \in \mathbf{H}_{i_4}(y_4) = \mathbf{H}_2(y_4)$, $y_4 \in \mathbf{H}_{i_5}(y_5) = \mathbf{H}_1(t)$]. We first show that $\overline{\mathcal{M}}_J^k$ is (weakly) finer than \mathbb{H}_J^k , that is, $\forall t \in T^k$, $\overline{\mathbf{M}}_J(t) \subseteq \mathbf{H}_J(t)$. Let $v \in \overline{\mathbf{M}}_J(t)$ and let $\langle y_1, y_2, \dots, y_m \rangle$ and $\langle i_2, \dots, i_m \rangle$ be sequences (of nodes and players, respectively) with the above-stated property. Since $y_j \in \mathbf{H}_{i_{j+1}}(y_{j+1})$, there is a sequence of nodes $\langle x_j, \dots, x_{j_{m(j)}} \rangle$ in T^k (which, w.l.o.g., we can assume to be distinct) such that, for all $r = 1, \dots, m(j)-1$, $\emptyset \neq \mathcal{H}_{i_{j+1}}^r(x_j) \cap \mathcal{H}_{i_{j+1}}^r(x_{j_{r+1}})$, $x_{j_1} = y_j$, $x_{j_{m(j)}} = y_{j+1}$. Since $\forall i_{j+1}, \forall w$, $\mathcal{H}_{i_{j+1}}^r(w) \subseteq \mathcal{H}_{i_j}^r(w)$, the sequence $v = y_1 = x_{1_1}, \dots, x_{1_{m(1)}} = y_2 = x_{2_1}, \dots, x_{2_{m(2)}} = y_3, \dots, y_m = t$ is such that each pair of consecutive elements is \sim_J -related. Therefore, $v \sim_J^* t$,
or
 $v \in \mathbf{H}_J(t)$.

Now we show that \mathbb{H}_J^k is (weakly) finer than $\overline{\mathcal{M}}_J^k$, that is, $\forall t \in T^k, H_J(t) \subseteq \overline{\mathcal{M}}_J(t)$. Let $v \in H_J(t)$. Then there exist sequences $\langle t_1, \dots, t_m \rangle$ and $\langle h_2, \dots, h_m \rangle$ of nodes and information sets, respectively, such that, $t_1 = v, t_m = t$, (w.l.o.g.) $t_j \neq t_{j+1}$, for all $j = 1, \dots, m, t_j \in T^k$, and, for all $j = 1, \dots, m-1, h_{j+1} \in \mathcal{H}_j^f(t_j) \cap \mathcal{H}_j^f(t_{j+1})$. Let $i_{j+1} \in J$ be the player moving at h_{j+1} . Then $h_{j+1} \in \mathcal{H}_{i_{j+1}}^f(t_j) \cap \mathcal{H}_{i_{j+1}}^f(t_{j+1})$, which implies that $t_j \sim_{i_{j+1}} t_{j+1}$. Therefore the sequences $\langle t_1, t_2, \dots, t_m \rangle$ and $\langle i_2, \dots, i_m \rangle$ (of nodes and players, respectively) are such that $\{i_2, \dots, i_m\} \subseteq J, t_1 = v, t_m = t$, and, for all $j = 1, \dots, m-1, t_j \in H_{i_{j+1}}(t_{j+1})$, that is, $v \in \overline{\mathcal{M}}_J(t)$. ■

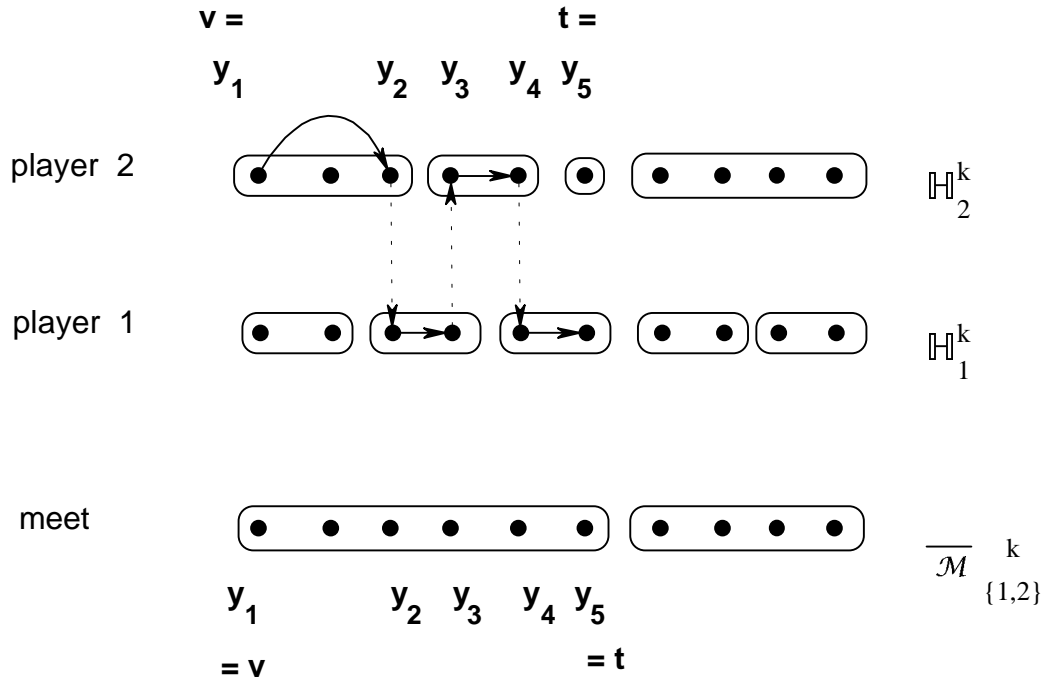


FIGURE 8

EXAMPLE 3. Consider the game of Figure 1. Let us focus on stage 2. In order to find the meet $\overline{\mathcal{M}}_{\{1,2,3\}}^2$ of the stage-2 maximum information partitions $\overline{\mathcal{K}}_1^2$, $\overline{\mathcal{K}}_2^2$ and $\overline{\mathcal{K}}_3^2$ we can either compute them first and then calculate their meet (as we did in section 4) or –using Proposition 4 –we can compute it directly using the relation \sim_J^* defined above for $J = N = \{1,2,3\}$. This is quickly done as follows: t_2 , t_3 and t_4 must all belong to the same cell, since they belong to an information set of one of the players, namely player 2. Furthermore, t_4 and t_6 must belong to the same cell since there is an information set of one of the players, namely player 3, that is crossed by a path starting at t_4 and also by a path starting at t_6 . On the other hand, since t_7 is a terminal node, its cell is the singleton $\{t_7\}$. Thus $\overline{\mathcal{M}}_{\{1,2,3\}}^2 = \{ \{t_2, t_3, t_4, t_6\}, \{t_7\} \}$.

6. Final remarks and conclusion

Restricting attention to the class of extensive forms defined by von Neumann and Morgenstern (1944) with the added assumption of perfect recall, we specified the information of each player at each node of the game-tree in a way which is coherent with the original information structure of the extensive form. We showed that this approach provides a framework for a formal and rigorous treatment of questions of knowledge and common knowledge at every node of the tree. We constructed a particular information completion and showed that it captures the notion of maximum information in the sense that it is the finest within the class of information completions that satisfy four natural properties. Using this notion of “maximum information” we were able to provide an alternative characterization of the meet of the information partitions. We also showed that, with maximum information, it is indeed true (as normally argued at an informal level) that at any node in a subgame it is common knowledge among all the players that the play of the game has reached that subgame. However, in general –that is, with less than maximum information – this statement is false (as a matter of fact the weaker statement that “all players know that the subgame has been reached” is false in general). On the other

hand it is always true (that is, for arbitrary information completions) that in simultaneous games at every decision node there are no non-trivial events that are common knowledge among all the players.

Throughout the paper attention was restricted to vN-M extensive forms. The reason for this is that in games that do *not* have this multi-stage structure, it may be problematic to define, for every player, a partition of the set of nodes that satisfies the four natural properties of Section 3. Consider, for example, the extensive form of Figure 9. Suppose that we want to construct an information partition for player 2. If we want the partition to satisfy property (1) (coherence with the information structure) then it must be that $[x_1]_2 = [x_2]_2 = \{x_1, x_2\}$ and $[x_4]_2 = [x_5]_2 = \{x_4, x_5\}$. By Property (2) (players remember their own past choices), $x_0 \notin [x_3]_2$. But this means that $[x_3]_2 = \{x_3\}$, that is, if node x_3 is reached, player 2 is informed. Then she will be able to discriminate between x_4 and x_5 depending on whether or not she received the information $\{x_3\}$. But this goes against the notion of information set. Alternatively, one could weaken Property (1) from equality to inclusion: if $t \in h \in \mathbb{H}_i$ then $[t]_i \supseteq h$. Following this route one would have to choose between $x_3 \in [x_1]_2$ and $x_3 \in [x_4]_2$. In the former case Property (2) would be violated (players remember their own past choices). In the latter case player 2 also needs to be told whether or not she has to move. If she is told that it is not her turn to move, then she will be able to deduce that node x_3 has been reached, leading to her ability to distinguish between x_4 and x_5 later on. All these problems can be eliminated by adding a dummy node and a dummy player with only one choice half way along the arc from x_1 to x_4 , that is, by transforming the game into a vN-M one.

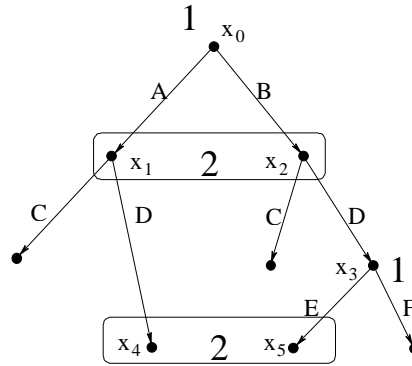


FIGURE 9

Prompted by a referee we conclude by discussing the connection between our notion of extended information structure and the notion of “normal form information set” introduced by Mailath, Samuelson and Swinkels (1993, 1994) (from now on MSS). Every information set h for any player i in a perfect recall game corresponds to the Cartesian product $S_i(h) \times S_{-i}(h)$ of the strategy profiles of the reduced normal form inducing a path through h . MSS identify the crucial structural property of such sets and propose to consider any set of strategy profiles of a reduced normal form game satisfying this property as a normal form information set. Of course, in general not all the normal form information sets of a given player correspond to actual information sets of the original extensive form. For example, the whole set of strategy profiles of the reduced normal form is a normal form information set for every player, but only one player moves at the root of the original extensive game. Analogously, we define sets of nodes which represent the information of a given player at any point of an extensive game and need not correspond to his actual information sets. For example, the singleton containing the root of the game is an “extended information set” for every player, although only one moves at the root. The obvious question is then: What is the relationship between extended information sets and normal form information sets? The answer is simple and does not depend on the precise mathematical definition given by MSS: an extended information set for player i always corresponds to a normal form information for i in the reduced normal form of the extensive game. In fact, let

t be a node at which player i does not move and consider the extended information set $H_i(t)$. Now construct a modified extensive game in which an additional stage is added, just before the stage $\ell(t)$ containing t . The nodes in this stage are just a copy of those in $\ell(t)$, but formally are decision nodes of player i . However, player i has a unique action at every such node leading to the corresponding node in stage $\ell(t)$ and his information sets are a copy of i 's extended information sets in stage $\ell(t)$. It is obvious from this construction that the modified game must have the same reduced normal form as the original one and that the strategy profiles inducing a path through $H_i(t)$ are precisely those which induce a path through the information set of the modified game containing the copy of node t . Since this is a proper information set (although a trivial one), the corresponding set of strategy profiles must be a normal form information set. (However, since player i does not move at t , the normal form information set corresponding to $H_i(t)$ is not "strict" in the sense of MSS, 1994, Definition 8.) On the other hand, it is easy to produce examples of normal form information sets which do not correspond to any extended information sets of the original extensive form game (see, for example, Figures 1 and 2 in MSS, 1994).

APPENDIX

The notation and terminology used in this paper are quite standard. For the reader's convenience we provide a list below.

<i>Notation</i>	<i>Terminology</i>
$N = \{1, \dots, n\}$	Set of personal players (that is, excluding nature)
T	Set of nodes
Z	Set of terminal nodes
\prec	Strict precedence relation on T : $t \prec t'$ if and only if $t \neq t'$ and t is a predecessor of t' .
\preceq	Weak precedence relation on T : $t \preceq t'$ if either $t = t'$ or $t \prec t'$
t_0	Root of the tree
$p(t)$ ($t \in T \setminus \{t_0\}$)	Immediate predecessor of node t
$S(t)$	Set of immediate successors of node t
$S(h, c)$	Set of immediate successors of nodes in information set h following the edges that constitute choice c (at h).
$\ell(t)$	Number of predecessors of node t (thus $\ell(t_0) = 0$ and, for every $t \neq t_0$, $\ell(t) = \ell(p(t)) + 1$)
T^k ($k \in \mathbb{N}$)	Set of "stage- k (or layer- k) nodes": $T^k = \{t \in T : \ell(t) = k\}$
\mathbb{H}_i ($i \in N$)	Set of player i 's information sets
$H(t)$ ($t \in T \setminus Z$)	Information set to which node t belongs

* Princeton University and University of California Davis, respectively. The authors benefited from comments by an anonymous referee.

¹ See, for example, Aumann (1976), Aumann and Brandenburger (1995), Bacharach (1985), Bonanno (1996), Geanakoplos (1992), Halpern (1986), Halpern and Moses (1992), Lismont (1993), Lismont and Mongin (1994), Milgrom (1981). (This is not meant to be an exhaustive list of relevant references.)

² See Bernheim, Peleg and Whinston (1987), Benoit and Krishna (1993), Farrell and Maskin (1989) and Greenberg (1990).

³ More precisely, the set of states (or possible worlds) would be of the form $T \times S$, where T is the set of nodes and S is some other set.

⁴ An extensive form is an extensive game without the payoff functions.

⁵ Thus, if t_0 denotes the root of the tree, $\ell(t_0) = 0$. The Appendix contains a complete list and explanation of the notation and terminology used in this paper, which are quite standard.

⁶ Recall that, according to the definition of extensive game (see, for example, Selten, 1975), the set of decision nodes of player i is partitioned into information sets (of player i). Throughout this paper we use

the expression “information set” to refer to these objects. On the other hand, the elements of \mathcal{K}_i (among which are i 's information sets) will always be called *cells*.

⁷ By Property (4), for every player i and every stage k , there is a subset of \mathcal{K}_i , call it \mathcal{K}_i^k , which is a partition of T^k . Thus for every i and every k we can define a *stage- k knowledge operator* $K_i^k: \mathcal{P}(T^k) \rightarrow \mathcal{P}(T^k)$ as follows: $K_i^k E = \{t \in T^k \mid [t]_i \subseteq E\}$. Similarly, for every stage k there is a subset of \mathcal{M} , call it \mathcal{M}^k , which is a partition of T^k . Thus we can define a *stage- k common knowledge operator* $CK^k: \mathcal{P}(T^k) \rightarrow \mathcal{P}(T^k)$ as follows: $CK^k E = \{t \in T^k \mid [t]_{\mathcal{M}^k} \subseteq E\}$. Note, however, that for every $E \subseteq T$, $K_i^k(E \cap T^k) = K_i E \cap T^k$ and, similarly, $CK^k(E \cap T^k) = CK E \cap T^k$.

⁸ This is the information at terminal nodes which is assumed in Fudenberg and Levine's (1993) notion of self-confirming equilibrium. In order to define more general notions of conjectural equilibria, it may be interesting to consider coarser information about terminal nodes. In particular, the one obtained by the coarsest information function, taking into account actual payoff information (see Battigalli, 1987). Furthermore, our analysis can easily be modified in order to take into account given information partitions on the set of terminal nodes (of course, the corresponding maximum information completion would not necessarily satisfy the property about subgames stated in Proposition 2).

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