Risk attitude and the shape of the von Neumann-Morgenstern utility-of-money function

Recall that the definition of risk aversion/neutrality/love applies only to money lotteries, where the basic outcomes are sums of money. Consider an individual whose preferences over money lotteries satisfy the axioms of expected utility theory. Then by the Expected Utility Theorem, there is a von Neumann-Morgenstern utility function $U(m)$ (where $m$ denotes money) that represents those preferences.

Now we show that if the individual is risk averse, $U(m)$ is a strictly concave function. The definition of strictly concave function is as follows.

**Definition.** A function $U(m)$ is strictly concave if for every two numbers $m_1$ and $m_2$ such that $m_1 \neq m_2$ and for every number $t$ strictly between 0 and 1,

$$U(t m_1 + (1-t) m_2) > t U(m_1) + (1-t) U(m_2).$$

The geometric interpretation is as follows. Given two points $A$ and $B$ in the Cartesian plane, for every number $t$ between 0 and 1, we can find the point $C(t) = t A + (1-t) B$. This is a point on the line segment joining $A$ and $B$. Letting $t$ vary from 0 to 1, we trace out the line segment joining $A$ and $B$. Thus, for example, $C(0)$ is point $B$. $C(1)$ is point $A$. $C(1/2)$ is the midpoint between $A$ and $B$, etc. For example, if $A = (20, 6)$ and $B = (10, 30)$ then the mid-point between $A$ and $B$ on the line segment joining $A$ and $B$ is $C(1/2) = \frac{1}{2} (20, 6) + \frac{1}{2} (10, 30) = \left( \frac{1}{2} 20 + \frac{1}{2} 10, \frac{1}{2} 6 + \frac{1}{2} 30 \right) = (15, 18)$.

Thus the definition of strict concavity has the following geometric interpretation:

![Graphical representation of concavity](image)

In light of this, for the case of an individual who satisfies the axioms of expected utility theory we can restate the definition of risk aversion as follows:
the individual is risk averse if, for any money lottery $L$, the utility of the expected value of $L$ is higher than the expected utility of $L$:

$$U(EV(L)) > EU(L).$$

For example, suppose that $U(m) = \sqrt{m}$. Then the individual is risk averse, because the utility function is strictly concave:

Consider the lotteries $L = \begin{pmatrix} \$0 & \$121 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $M = \begin{pmatrix} \$25 & \$100 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$. Then

$$EU(L) = \frac{1}{2} \sqrt{0} + \frac{1}{2} \sqrt{121} = \frac{\sqrt{121}}{2} = 5.5$$

and

$$EU(M) = \frac{1}{5} \sqrt{25} + \frac{4}{5} \sqrt{100} = \frac{5}{2} + \frac{2}{5} 10 = 7$$

thus the individual would prefer $M$ to $L$. On the other hand, since $EV(L) = \frac{1}{2} 0 + \frac{1}{2} 121 = 60.5$ while $EV(M) = \frac{1}{5} 25 + \frac{4}{5} 100 = 15 + 40 = 55$, a risk neutral person would prefer $L$ to $M$.

The von Neumann-Morgenstern utility-of-money function of a risk-neutral person is a straight line, that is, it is of the form $U(m) = a + bm$ with $b > 0$:

Starting with $U(m) = a + bm$ we can normalize the utility function by first subtracting $a$ (thus getting $\tilde{U}(m) = bm$) and then dividing by $b$, thus ending up with the identity function $V(m) = m$. Using the identity function it turns out that the expected utility of a money lottery coincides with the expected value, confirming that a risk-neutral person ranks money lotteries according to their expected value.

The von Neumann-Morgenstern utility-of-money function of a risk-loving person is convex: