A binary lottery is a lottery with only two prizes. Call the two prizes y and z. Let p be the probability of prize y and \((1 - p)\) the probability of prize z. Thus a binary lottery has the following form:

<table>
<thead>
<tr>
<th>outcome or prize</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability</td>
<td>p</td>
<td>(1 - p)</td>
</tr>
</tbody>
</table>

We shall assume that the prizes y and z are sums of money. We shall consider p and \((1 - p)\) as fixed numbers and allow y and z to vary. Thus a binary lottery can be identified with a point in the Cartesian plane \((y,z)\). If \(y = z\) then the lottery \((y,y)\) represents the situation where you get \(y\) with probability \(p\) and \(y\) with probability \((1 - p)\), that is, you get \(y\) for sure.

Consider an individual whose utility of money function is \(U(m)\). We assume that \(U'(m) > 0\) for every \(m\), that is, that the individual prefers more money to less. Given a lottery \((y,z)\), the individual’s expected utility is given by:

\[
p U(y) + (1 - p) U(z).
\]

Given two lotteries \(A = (y_1, z_1)\) and \(B = (y_2, z_2)\), the individual will prefer A to B if

\[
EU(A) = p U(y_1) + (1 - p) U(z_1) > EU(B) = p U(y_2) + (1 - p) U(z_2),
\]

will prefer B to A if the above inequality is reversed and will be indifferent between A and B if \(EU(A) = EU(B)\). For example, if \(p = \frac{1}{4}\) and \(U(m) = 2m\), then the individual will be indifferent between \((100, 80)\) and \((130, 70)\) and \((85, 85)\): his expected utility is 170 in all three cases.

For any given lottery in the Cartesian \((y,z)\) plane, there will be an indifference curve through it. We want to relate the shape of the indifference curves of the individual to his attitude towards risk.

### CASE 1: RISK NEUTRALITY

If the individual is risk-neutral, his utility function is a straight line and, therefore, is of the form \(V(m) = am + b\), with \(a > 0\). If we multiply \(V(m)\) by \(\frac{1}{a}\) and subtract \(\frac{b}{a}\) we obtain a new utility function \(U(m) = m\), which is an alternative representation of the same preferences. This is an important thing to remember: the preferences of a risk-neutral person can always be represented by the function \(U(m) = m\), so that the expected utility of a lottery coincides with the expected value.

Fix an arbitrary lottery \(A = (y,z)\). Let us try to find another lottery \(B = (y + \Delta y, z + \Delta z)\) that lies on the same indifference curve. Then it must be \(EU(A) = EU(B)\):

\[
p y + (1 - p) z = p (y + \Delta y) + (1 - p) (z + \Delta z)
\]

The RHS is equal to

\[
p y + (1 - p) z + p \Delta y + (1 - p) \Delta z
\]

Thus, \(EU(A) = EU(B)\) requires

\[
p \Delta y + (1 - p) \Delta z = 0
\]

that is,

\[
\frac{\Delta z}{\Delta y} = -\frac{p}{1 - p}
\]

Thus the indifference curves are straight lines with slope \(-\frac{p}{1 - p}\).
In the above figure, going from A to B we have: \( \Delta y = -4 \) and, since \( p = \frac{1}{3} \), 
\[ -\frac{p}{1-p} = -\frac{1}{2} \]. Thus to remain on the same indifference curve we must have
\[ \Delta z = \left( -\frac{p}{1-p} \right) \Delta y = \left( -\frac{1}{2} \right) (-4) = 2. \]
That is, \( z \) must increase from 8 to 10.

### CASE 2: RISK AVERSION

If the individual is risk averse, \( U(m) \) is a strictly concave function: for every two numbers \( m_1 \) and \( m_2 \) such that \( m_1 \neq m_2 \) and for every number \( t \) strictly between 0 and 1,
\[ U(t m_1 + (1-t) m_2) > t U(m_1) + (1-t) U(m_2). \]

Now we want to show that for a risk-averse individual the indifference curves in the \((y,z)\) Cartesian product are convex towards the origin as shown in the figure below:
We can prove this by showing that if we take an arbitrary indifference curve and two arbitrary points A and B on it, all the points on the line segment joining A and B (apart from A and B themselves) lie above the indifference curve. This is shown in the following figure. Suppose that A=(y,z) and B=(y',z') lie on the same indifference curve, that is, EU(A) = EU(B), where

\[ EU(A) \equiv p \, U(y) + (1-p) \, U(z) \quad \text{and} \quad EU(B) \equiv p \, U(y') + (1-p) \, U(z'). \]

Fix an arbitrary number \( t \) strictly between 0 and 1 and consider the point

\[ C = t \, A + (1-t) \, B = \left( ty+(1-t)y', \, tz+(1-t)z' \right) \]

on the line segment joining A and B. We want to show that point C lies on a higher indifference curve, that is,

\[ EU(C) > EU(A) = EU(B). \]

Now,

\[ EU(C) \equiv p \, U\left( ty+(1-t)y' \right) + (1-p) \, U\left( tz+(1-t)z' \right). \] (1)

By definition of strict concavity,

\[ U\left( ty+(1-t)y' \right) > t \, U(y) + (1-t) \, U(y') \] (2)

and

\[ U\left( tz+(1-t)z' \right) > t \, U(z) + (1-t) \, U(z') \] (3)

Multiplying both sides of (2) by \( p \) and both sides of (3) by \( 1-p \) and adding the resulting inequalities we obtain

\[ pU\left( ty+(1-t)y' \right) + (1-p)U\left( tz+(1-t)z' \right) > p[t \, U(y) + (1-t) \, U(y')] + (1-p)[t \, U(z) + (1-t) \, U(z')]. \]

The LHS is \( EU(C) \), while the RHS is equal to

\[ t \left[ pU(y) + (1-p)U(z) \right] + (1-t) \left[ pU(y') + (1-p)U(z') \right] = t \, EU(A) + (1-t) \, EU(B) = EU(A) \]

since \( EU(A) = EU(B) \). Thus we have proved that \( EU(C) > EU(A) \).
CASE 3 : RISK LOVING

If the person is risk-loving then his indifference curves will be concave towards the origin (the proof is similar).

The slope of an indifference curve

Fix a point \( A = (y,z) \). We want to know what the slope of the indifference curve through \( A \) is. Fix an arbitrary point \( B = (y+dy, z+dz) \). If \( dy \) and \( dz \) are small, \( B \) will be close to \( A \).

The expected utility of lottery \( A \) is:
\[
EU(A) = p \, U(y) + (1 - p) \, U(z).
\]
The expected utility of \( B \) is:
\[
EU(B) = p \, U(y+dy) + (1 - p) \, U(z+dz).
\]

Brief revision of calculus: meaning of the derivative. Given a function \( y = f(x) \), let \( a \) be a number and suppose that you know the value \( f(a) \). Let \( d \) be a number (positive or negative) and consider the number \( a + d \). What is the value of \( f(a+d) \)? If \( d \) is small (so that \( a+d \) is close to \( a \)), then \( f(a+d) \) is approximately equal to \( f(a) + f'(a) \cdot d \). For example, let \( f(x) = \sqrt{10x + \frac{x^2}{2}} \). Let \( a = 90 \). \( f(90) = 4080 \). \( f'(90) = \frac{10}{2\sqrt{900}} + 90 = 90.167 \). Let \( d = 0.06 \), so that \( a + d = 90.06 \). \( f(90.06) = 4085.4127 \). On the other hand, \( f(90) + f'(90) \cdot (0.06) = 4085.4109 \). Pretty close! The smaller \( d \), the closer you get to the true value.

Applying the above to \( U(y+dy) \) and to \( U(z+dz) \) we get:
\[
U(y+dy) = U(y) + U'(y) \, dy \quad \text{and} \quad U(z+dz) = U(z) + U'(z) \, dz.
\]

Thus
\[
EU(B) = p \left[ U(y) + U'(y) \, dy \right] + (1 - p) \left[ U(z) + U'(z) \, dz \right]
\]
\[
= p \, U(y) + (1 - p) \, U(z) + [p \, U'(y) \, dy + (1 - p) \, U'(z) \, dz]
\]
\[
= EU(A) + [p \, U'(y) \, dy + (1 - p) \, U'(z) \, dz]
\]

Now, if \( A \) and \( B \) are on the same indifference curve, then \( EU(A) = EU(B) \) (by definition of indifference curve). This implies that
\[
[p \, U'(y) \, dy + (1 - p) \, U'(z) \, dz] = 0,
\]
which gives:
\[
\frac{dz}{dy} = -\frac{p}{1 - p} \frac{U'(y)}{U'(z)}
\]

The slope of the indifference curve through point \( A = (y,z) \) is equal to
\[
-\frac{p}{1 - p} \frac{U'(y)}{U'(z)}
\]