Let us go back the case of an individual who has an initial wealth of $W$ and with probability $p$ ($0 < p < 1$) he faces a loss of $x$ ($0 < x < W$). His initial situation can be represented as a point in a two-dimensional diagram where we measure on the horizontal axis his wealth in the bad state (loss), denoted by $W_1$, and on the vertical axis his wealth in the good state (no loss), denoted by $W_2$. An insurance contract takes the individual from the initial point of no insurance to some other point.

An insurance contract can be described by a pair $(h, D)$, where $h$ is the premium and $D$ is the deductible. Note that the premium is paid in any case, that is both in the good state and in the bad state. Given a contract $(h, D)$, wealth in the good state will be $(W - h)$ and wealth in the bad state will be $W - h - D$. Thus if we represent a contract in the $(W_1, W_2)$ plane, then the premium is $(W - W_2)$ and the deductible is $(W_2 - W_1)$. For example, in Figure 1 Contract $A$ involves a premium of $5,000 - 4,500 = $500 and a deductible equal to $4,500 - 3,900 = $600. and contract $B$ involves a premium of $(5,000 - 4,100) = $900 and zero deductible, that is, full insurance.

![Figure 1](image)

**ISOPROFIT LINES**

The expected profit from insurance contract $(h, D)$ is

$$h - p(x - D) = [h - px + pD]$$ \hspace{1cm} (1.a)

If we represent the contract as a point in the $(W_1, W_2)$ plane, then the expected profit from the contract can be written as

$$\frac{(W - W_2) - px + p(W_2 - W_1)}{h} = \frac{W - px - (1 - p)W_2 - pW_1}{W_2 - W_1}$$ \hspace{1cm} (1.b)

An *isoprofit line* is a line that joins all the contracts that yield the same expected profit. Suppose that $A = (W_1^A, W_2^A)$ and $C = (W_1^C, W_2^C)$ are two contracts that yield the same profit, that is, $\pi(A) = \pi(C)$. The slope of the line joining these two contracts is: $\frac{\text{rise}}{\text{run}} = \frac{W_2^A - W_2^C}{W_1^A - W_1^C}$. Now, from
(1.b) we get that \( \pi(A) = W - px - (1-p)W_A - pW_A \) and \( \pi(C) = W - px - (1-p)W_C - pW_C \). Thus by setting \( \pi(A) = \pi(C) \) we get: 

\[
-(1-p) \left( W_A - W_C \right) = p \left( W_A - W_C \right)
\]

so that

\[
\frac{\text{rise}}{\text{run}} = \frac{W_A - W_C}{W_A - W_C} = -\frac{p}{1-p}
\]

Hence the isoprofit line through points \( A \) and \( C \) is a straight line with slope \(-\frac{p}{1-p}\).

Recall that points below an isoprofit line correspond to contracts with a higher expected profit:

Thus moving from \( A \) inside the shaded area profits increase (convex combination of no change and increase). The isoprofit line that goes through the No Insurance point \((W - x, W)\) is the zero profit line.
Fist we want to draw the indifference curves of the consumer in the \((W_1,W_2)\) plane. Let \(U(m)\) be the utility-of-money function and assume that \(U' > 0\) and \(U'' < 0\), that is, the consumer is risk averse. Consider a point \(A = (W_1^A,W_2^A)\) and let \(C = (W_1^C,W_2^C)\) be a point close to \(A\) and suppose that \(A\) and \(C\) lie on the same indifference curve. Then \(EU(A) = pU(W_1^A) + (1-p)U(W_2^A)\) and \(EU(A) = EU(C)\). If \(C\) is close to \(A\), then \(W_1^C\) is close to \(W_1^A\) so that

\[
U(W_1^C) = U(W_1^A) + U'(W_1^A) \times (W_1^C - W_1^A) \quad (2.a)
\]

Similarly, \(W_2^C\) is close to \(W_2^A\) so that

\[
U(W_2^C) = U(W_2^A) + U'(W_2^A) \times (W_2^C - W_2^A) \quad (2.b)
\]

Hence,

\[
EU(C) = pU(W_1^C) + (1- p)U(W_2^C) \approx p \left[U(W_1^A) + U'(W_1^A) \times (W_1^C - W_1^A)\right] + (1- p) \left[U(W_2^A) + U'(W_2^A) \times (W_2^C - W_2^A)\right] =
\]

\[EU(A) + pU'(W_1^A) \left(W_1^C - W_1^A\right) + (1- p)U'(W_2^A) \left(W_2^C - W_2^A\right)\]

Since \(EU(A) = EU(C)\), it follows that \(pU'(W_1^A) \left(W_1^C - W_1^A\right) + (1- p)U'(W_2^A) \left(W_2^C - W_2^A\right) = 0\).

Thus \(\frac{\text{rise}}{\text{run}} = \frac{W_2^C - W_2^A}{W_1^C - W_1^A} = -\frac{p}{1- p} \frac{U'(W_1^A)}{U'(W_2^A)}\). Hence

\[
\text{The slope of the indifference curve at point } (W_1,W_2) \text{ is} \quad -\frac{p}{1- p} \frac{U'(W_1^A)}{U'(W_2^A)}.
\]

By strict concavity of the utility function, if \(m_1 < m_2\) then \(U'(m_1) > U'(m_2)\). Thus

- Above the \(45^\circ\) line (where \(W_1 < W_2\)), the slope of the indifference curve is greater in absolute value than the slope of the isoprofit line that goes through that point, that is,

\[
\frac{p}{1- p} \frac{U'(W_1^A)}{U'(W_2^A)} > \frac{p}{1- p}
\]

This is because \(W_1 < W_2\) (we are above the \(45^\circ\) line) and thus \(U'(W_1) > U'(W_2)\), so that \(\frac{U'(W_1^A)}{U'(W_2^A)} > 1\).

- Along the \(45^\circ\) line (where \(W_1 = W_2\)), the slope of the indifference curve is equal to the slope of the isoprofit line that goes through that point, that is,
Putting together what we found above, namely that

(1) at a point $A$ above the $45^\circ$ line the slope of the indifference curve is greater in absolute value than the slope of the isoprofit line that goes through that point, and

(2) moving away from a point $A$ above the $45^\circ$ line in the direction between the vertical direction downwards and the direction of the isoprofit line (which has slope $-\frac{p}{1-p}$), profits increase.

Thus profits for the insurance company increase as we move along the indifference curve that goes through point $A$ towards the $45^\circ$ line, as shown in the following figure.

![Figure 5](image-url)

What is the maximum premium, call it $h^*$, that an individual would be willing to pay for full insurance? It is the solution to the equation

$$U(W - h) = pU(W - x) + (1 - p)U(W)$$

For example, if $W = 1,600$, $x = 700$, $p = \frac{1}{10}$ then $h^*$ is given by the solution to

$$\sqrt{1,600 - h} = \frac{1}{10} \sqrt{1,600 - 700} + \frac{9}{10} \sqrt{1,600} = 39$$

which is $h^* = 79$.

Next we show that $px < h^*$. To begin with, note that, by definition of risk-aversion, the expected utility of a lottery is less than the utility of the expected value:
By definition of $h^*$, 
\[ U(W - h^*) = pU(W - x) + (1 - p)U(W) \]  
Thus from (1) and (2) we get 
\[ U(W - h^*) < U(W - px). \]  
Since $U$ is increasing, it follows that 
\[ W - h^* < W - px \]  
that is, 
\[ px < h^*. \]  

Next we show that $h^* < x$. First of all, note that 
\[ U(W - x) = pU(W - x) + (1 - p)U(W - x) < pU(W - x) + (1 - p)U(W) = U(W - h^*) \]  
That is, 
\[ U(W - x) < U(W - h^*) \]  
Which implies, since $U$ is increasing, that 
\[ h^* < x. \]  

This can be shown graphically as follows.

Note: 
1. $h^* < x$
2. $h^* > px$, i.e. risk-averse person willing to pay a premium which is higher than the expected loss.

Figure 6

Two observations:
1. We have assumed that the granting of full insurance does not affect the value of \( p \) (the probability of loss). This, however, is often not true. If you are insured, you don’t face a risk and therefore you exert less effort or care in trying to prevent a loss (e.g. you are less careful about locking your bike or locking the front door of your house).

2. The fact that \( h^* > px \) is what makes the sale of insurance policies a profitable business. If the probability of losses is independent across consumers and the insurance company sells a large number \( N \) of policies, then, by the law of large numbers in probability theory, it will have to make payments of \( $pxN \); however, it will be collecting \( $h^*N \), thus making a profit of \( $(h^* - px)N \) (assuming no other costs: an assumption that we will relax later).

**What happens to \( h^* \) if \( p \) increases?** For example, consider a probability of loss \( q > p \).

Thus \( \frac{dh^*}{dp} > 0 \).
CHOOSING A CONTRACT FROM A MENU

Often insurance companies offer a menu of possible contracts, not just one contract. Typically consumers have a choice between a higher premium and higher coverage or a lower premium and lower coverage.

EXAMPLE. Consider an individual whose initial wealth is $1,000. He faces a potential loss of $400, with probability $\frac{1}{5}$. Suppose that the insurance company offers the following options:

<table>
<thead>
<tr>
<th>premium</th>
<th>deductible</th>
</tr>
</thead>
<tbody>
<tr>
<td>contract 1</td>
<td>$82</td>
</tr>
<tr>
<td>contract 2</td>
<td>$62</td>
</tr>
<tr>
<td>contract 3</td>
<td>$40</td>
</tr>
</tbody>
</table>

The individual’s utility-of-money function is $U(m) = \sqrt{m}$. Which contract would he choose?

No insurance:

\[
\begin{pmatrix}
1000 & 600 \\
0.8 & 0.2 \\
\end{pmatrix}
\]

Expected Utility

\[
0.8\sqrt{1000} + 0.2\sqrt{600} = 30.197
\]

Full insurance:

\[
\begin{pmatrix}
1000 - 82 \\
1 \\
\end{pmatrix} = \begin{pmatrix}
918 \\
1 \\
\end{pmatrix}
\]

\[
\sqrt{918} = 30.299
\]

Decuctible of 100:

\[
\begin{pmatrix}
1000 - 62 & 1000 - 100 - 62 \\
0.8 & 0.2 \\
\end{pmatrix} = \begin{pmatrix}
938 & 838 \\
0.8 & 0.2 \\
\end{pmatrix}
\]

\[
0.8\sqrt{938} + 0.2\sqrt{838} = 30.291
\]

Decuctible of 200:

\[
\begin{pmatrix}
1000 - 40 & 1000 - 200 - 40 \\
0.8 & 0.2 \\
\end{pmatrix} = \begin{pmatrix}
960 & 760 \\
0.8 & 0.2 \\
\end{pmatrix}
\]

\[
0.8\sqrt{960} + 0.2\sqrt{760} = 30.301
\]

Thus the best option is contract 3 (deductible of $200) and the next best option is full insurance.

Now suppose that the consumer can choose any contract from the set of contracts that yield zero profits. Then for each insurance policy the issuer collects premium $h$ and expects to pay out $p(x - D)$. Thus zero profits means that $h$ and $D$ are such that

\[
h = p(x - D).
\]

Suppose that customers are free to choose any premium-deductible combination $(h, D)$ that satisfies the above equation.

What premium-deductible combination $(h, D)$ would a risk-averse individual choose? Clearly, once you choose $D$, the premium is determined by the equation $h = p(x - D)$. Thus

Wealth if no loss: $W - h = W - p(x - D) = W - px + pD$
Wealth if loss: \[ W - h - D = W - p(x - D) - D = W - px + pD - D = W - px - (1-p)D \]

Thus expected utility from policy \((h,D)\) is

\[
f(D) = p U(W - px - (1-p)D) + (1-p) U(W - px + pD)
\]

The individual will choose \(D\) to maximize \(f(D)\). A necessary condition is \(f'(D) = 0\). Now

\[
f'(D) = p U'(W - px - (1-p)D) (-(1-p)) + (1-p) U'(W - px + pD) (p)
\]

Thus we need \(- (1-p) p U'(W - px - (1-p)D) + (1-p) p U'(W - px + pD) = 0\), that is,

\[
U'(W - px + pD) = U' \left( \frac{W - px - (1-p)D}{W - px + pD - D} \right)
\]

Since \(U\) is strictly concave, the slope of \(U\) is different at any two different points. Thus the only way to satisfy the equation \(U'(x) = U'(y)\) is to have \(x = y\). Thus to satisfy the above equation we need

\[
W - px + pD = W - px + pD - D
\]

and this requires \(D = 0\). Thus the individual would choose full insurance. [Of course we already knew that, because it is only along the 45\(^o\) line that there is a tangency between the indifference curve and the zero-profit line: on the 45\(^o\) line the slope of both is \(-\frac{p}{1-p}\)].

**Case 1: THE INSURANCE INDUSTRY IS A MONOPOLY**

The monopolist will want to offer a contract that lies on the indifference curve through the No Insurance point (if it offered a contract above that indifference curve it would not be maximizing its profits, because coming vertically down from that point to the indifference curve would yield an increase in profits: see Figure 7). Furthermore, as seen above, moving along the indifference curve towards the 45\(^o\) line profits increase. Thus the monopolist would want to offer full insurance at premium \(h^*\).
Case 2: FREE ENTRY in the Insurance Industry leads to ZERO PROFITS

A contract that yields zero profit is called a **fair contract** and the zero profit line is called the *fair odds* line. Recall that the zero profit line is the straight line that goes through the No Insurance point and has slope \(-\frac{p}{1-p}\).

**Define an equilibrium in a competitive insurance industry as a situation where every firm makes zero profits and no firm (existing or new) can make positive profits by offering a new contract.** What contract(s) will be offered in this industry?

First of all, we show that **at least one firm must offer the full-insurance contract** (given by \(h = px\) and \(D = 0\)). Suppose not. Then take a contract on the zero-profit line that is offered and that some consumers are buying. Since it is not the full-insurance contract, it must lie above the 45\(^\circ\) line, like contract \(A\) in Figure 9 below. Hence the slope of the indifference curve through \(A\) is steeper than the zero-profit line and, therefore, there is a contract \(B\) that lies below the zero-profit line (hence \(\pi(B) > 0\)) and above the indifference curve through \(A\), so that a firm that offered contract \(B\) would attract all the consumers who were buying \(A\) and make positive profit.

Secondly, since the full-insurance contract is offered by at least one firm, every consumer purchases that contract (because any other contract that is offered must be on the zero-profit line and, as just shown, it must lie on a lower indifference curve that the full insurance contract.)