OPTIMAL RISK-SHARING IN EMPLOYMENT CONTRACTS

Consider an employment contract between a Principal (the employer) and an Agent (the employee). The Agent is to perform a job whose outcome is not entirely under her control: it is partly affected by random factors. For example, the Principal could be the owner of a firm, the Agent the manager and the outcome the firm’s profits: no matter how hard the manager works, the firm’s profits might be low because of a recession or unexpected competition, etc. For simplicity we shall assume that there are only two possible outcomes: $x_1$ and $x_2$, with $x_1 > x_2$.

Let $p$ be the probability of outcome $x_1$ and $(1 - p)$ the probability of outcome $x_2$. Thus the Agent is hired to perform a job whose outcome is the following lottery:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
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<tbody>
<tr>
<td>Probability</td>
<td>$p$</td>
<td>$1 - p$</td>
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The Principal and the Agent have to agree on the payment to the Agent. In principle, since there are two possible outcomes, the payment to the Agent can be made dependent on the outcome. For example, if $x_1 = 1,000$ and $x_2 = 200$, the owner of the firm (Principal) and the manager (Agent) can sign a contract that says: “If the firm’s profits are $1,000, the manager will be paid $400, while if profits are $200 the manager will be paid $80”. Thus a contract can be thought of as a pair of numbers $(w_1, w_2)$, where $w_1$ is the Agent’s salary if the outcome is $x_1$, and $w_2$ is the Agent’s salary if the outcome is $x_2$ (in the above example, $w_1 = 400$, $w_2 = 80$). Of course, Principal and Agent can agree on a contract $(w_1, w_2)$ such that $w_1 = w_2$, that is, a contract that guarantees a fixed income to the Agent. Similarly, they could agree on a contract $(w_1, w_2)$ such that $x_1 - w_1 = x_2 - w_2$, so that the Principal knows for sure what his income will be. Going back to the above example, where $x_1 = 1,000$ and $x_2 = 200$, if Principal and Agent sign contract A = $(w_1 = 100, w_2 = 100)$ then the Agent will receive $100$ for sure, while the Principal will receive $(1,000 - 100) = 900$ with probability $p$ and $(200 - 100) = 100$ with probability $(1 - p)$. On the other hand, if they sign contract B = $(w_1 = 800, w_2 = 0)$ then the Principal will get $200$ for sure, while the Agent will get $800$ with probability $p$ and nothing with probability $(1 - p)$.

The set of all possible contracts can be represented by an Edgeworth box ($0_A$ is the origin for the Agent and $0_P$ is the origin for the Principal; the size of the horizontal side of the box is equal to $x_1$ and the size of the vertical side of the box is $x_2$). Every point in the box represents a possible contract.
Let $U(m)$ be the Principal’s utility-of-money function and $V(m)$ be the Agent’s utility-of-money function. Given a contract $C = (w_1, w_2)$, the Principal’s expected utility will be:

$$EU(C) \equiv p U(x_1 - w_1) + (1 - p) U(x_2 - w_2)$$

while the Agent’s expected utility will be:

$$EV(C) \equiv p V(w_1) + (1 - p) V(w_2).$$

We want to characterize the set of Pareto efficient contracts.

**Definition.** A contract $A$ is **Pareto dominated** if there is another contract $B$ such that:

- either $EU(B) > EU(A)$ and $EV(B) = EV(A)$
- or $EU(B) = EU(A)$ and $EV(B) > EV(A)$
- or $EU(B) > EU(A)$ and $EV(B) > EV(A)$
Definition. A contract that is not Pareto dominated is called **Pareto efficient** (or Pareto optimal). Thus contract C is Pareto efficient if for every other contract D, either
\[ \text{EU}(D) < \text{EU}(C) \]
or
\[ \text{EV}(D) < \text{EV}(C) \]
or both. That is, any alternative contract makes at least one of the two parties worse off.

Example. Suppose that \( x_1 = 1,000, \ x_2 = 600, \ p = \frac{1}{3} \) \( \left( \text{so that} \ 1 - p = \frac{2}{3} \right) \). Suppose also that the Principal’s utility function is given by
\[ U(m) = \sqrt{m} \]
while the Agent’s utility function is given by
\[ V(m) = 2m + 1. \]
Consider contract \( A = (w_1=400, w_2=400) \). This contract is not Pareto efficient: it is dominated, for example, by contract \( B = (w_1=676, w_2=276) \). In fact,
\[ \text{EU}(B) = \frac{1}{3} \sqrt{324} + \frac{2}{3} \sqrt{324} = 18 > \text{EU}(A) = \frac{1}{3} \sqrt{600} + \frac{2}{3} \sqrt{200} = 17.6 \]
and
\[ \text{EV}(B) = \frac{1}{3} [2(676)+1] + \frac{2}{3} [2(276)+1] = 819.67 > \text{EV}(A) = \frac{1}{3} [2(400)+1] + \frac{2}{3} [2(400)+1] = 801. \]
Thus it would be silly for Principal and Agent to sign contract A when they both prefer contract B.

How do we find the Pareto efficient contracts? First of all, we need to add indifference curves to our Edgeworth box. From the lecture on binary lotteries we know that the Principal’s indifference curves are convex to the origin if the Principal is risk-averse and are straight lines if the Principal is risk-neutral. In the following figure we show the case where the Principal is risk-averse.

![Direction of increasing utility for the Principal](image)

Similarly in the following figure we show the Agent’s indifference curves (we have assumed that the Agent is risk-averse).
Now, consider an arbitrary contract, such as S in the following figure. Draw the indifference curve of the Principal through S and the indifference curve of the Agent through S. If the two indifference curves cross, then the contract is **not** Pareto efficient: any contract in the shaded area Pareto dominates contract S.

Thus a necessary condition for a contract S to be Pareto efficient is that the indifference curve of the Principal through S and the indifference curve of the Agent through S be tangent, as shown in the following figure.
The two indifference curves are tangent at point $S$ if and only if they have the same slope at that point. We know from the lecture on binary lotteries that the slope of the indifference curve of the Principal at contracts $S = (w_1, w_2)$ is given by:

$$- \frac{U'(x_1 - w_1)}{U'(x_2 - w_2)} \frac{p}{1-p}$$

and the slope of the indifference curve of the Agent at $S = (w_1, w_2)$ is given by:

$$- \frac{V'(w_1)}{V'(w_2)} \frac{p}{1-p}$$

Thus a necessary condition for a contract $S = (w_1, w_2)$ to be Pareto efficient is that

$$\frac{U'(x_1 - w_1)}{U'(x_2 - w_2)} = \frac{V'(w_1)}{V'(w_2)}$$

We shall use the symbol $\heartsuit$ to refer to the above equation.

We can now examine a number of different possibilities.

**CASE 1: the Principal is risk-neutral, the Agent is risk-averse**

In this case, $U(m) = a \cdot m + b$ (with $a > 0$), while $V(m)$ is such that $V''(m) < 0$. Thus $U'(m) = a$ for every $m$. Hence

$$\frac{U'(x_1 - w_1)}{U'(x_2 - w_2)} = \frac{a}{a} = 1.$$ 

Hence equation $\heartsuit$ becomes

$$\frac{V'(w_1)}{V'(w_2)} = 1$$
that is, $V'(w_1) = V'(w_2)$. Since $V''(m) < 0$, the function $V'(m)$ is strictly decreasing in $m$ and therefore the equality $V'(w_1) = V'(w_2)$ can only be satisfied if $w_1 = w_2$. That is, the Agent must be guaranteed income certainty: her income cannot be contingent on the outcome. Thus the set of Pareto efficient contracts is the $45^\circ$ line out of the origin for the Agent. The indifference curves of the Agent are straight lines with slope $\frac{p}{1-p}$.

Claim. The set of Pareto efficient contracts is the red line.

Proof. Take a point on one of the sides but not on the thick lines, e.g. point $S$ shown below (for interior points, where $0 < w_1 < x_1$ and $0 < w_2 < x_2$, the proof is given above using the slopes of the indifference curves). Draw the indifference curve of the Principal through that point:
Since the slope of the Principal’s indifference curve is \(-\frac{p}{1-p}\) (recall that the Principal is risk-neutral), all contracts on this indifference curve have the same expected value. Thus contract T has the same expected value as S, but gives this amount to the Agent for sure. Hence the Agent prefers T to S by definition of risk-aversion. This argument takes care of all points except those on the thick vertical line in the figure below. For a point like V we have to use a different argument. Extend the 45° line.

Then draw the line of slope \(-\frac{p}{1-p}\) from point V to the extended 45° line. All points on the segment (V,W] are better than V because for a risk-averse person the indifference curves cannot cross this segment, since W is strictly preferred to V.

**CASE 2: the Principal is risk-averse, the Agent is risk-neutral**

In this case \(U(m)\) is such that \(U''(m) < 0\), while \(V(m) = a m + b\) (with \(a > 0\)). Then equation \(\heartsuit\) becomes:

\[
\frac{U'(x_1 - w_1)}{U'(x_2 - w_2)} = 1.
\]

As before, this equality can be satisfied only if \(x_1 - w_1 = x_2 - w_2\), that is, the **Principal must be given income certainty**. Thus the set of Pareto efficient contracts is the 45° line out of the origin for the Principal. The indifference curves of the Agent are straight lines with slope \(-\frac{p}{1-p}\).
The set of Pareto efficient contracts is the thick line.

**CASE 3: both Principal and Agent are risk-neutral**

In this case $U(m) = a \cdot m + b$ (with $a > 0$) and $V(m) = c \cdot m + d$ (with $c > 0$). The indifference curves of the Principal are straight lines with slope $-\frac{p}{1-p}$. The same is true for the Agent. Hence the indifference curve of Principal and the indifference curve of Agent through any contract coincide (and therefore are tangent). It follows that any contract is Pareto efficient.

**CASE 4: both Principal and Agent are risk-averse**

In this case both Principal and Agent would like income certainty but of course income certainty for both is impossible. To check if a specific contract is Pareto efficient one has to check equation $\mathcal{C}$ directly. For example, let $U(m) = \sqrt{m}$ and $V(m) = 82 - \left(10 - \frac{m}{100}\right)^2 - 1$. Let $x_1 = 800$ and $x_2 = 200$. Consider the contract $(w_1 = 400, w_2 = 100)$. Is it Pareto efficient? We have to check if equation $\mathcal{C}$ is satisfied. Now, $U'(m) = \frac{1}{2\sqrt{m}}$ and $V'(m) = \frac{2}{100} \left(10 - \frac{m}{100}\right)$. Thus

$$\frac{U'(x_1 - w_1)}{U'(x_2 - w_2)} = \frac{U'(400)}{U'(100)} = \frac{1}{2}$$

Principal

and

$$\frac{V'(w_1)}{V'(w_2)} = \frac{V'(400)}{V'(100)} = \frac{12}{18}$$

Agent.

Since $\frac{1}{2} \neq \frac{12}{18}$, the above contract is not Pareto efficient.
The indifference curve of the Principal is less steep than the indifference curve of the Agent at point \((w_1 = 400, w_2 = 100)\): see the following figure.

The grey area shows contracts that are Pareto superior to \((w_1 = 400, w_2 = 100)\). Thus to construct a Pareto superior contract we need to increase \(w_1\) and decrease \(w_2\). Recall that \(x_1 = 800\) (with probability \(\frac{1}{3}\)) and \(x_2 = 200\) (with probability \(\frac{2}{3}\)). Let \(S = (w_1 = 400, w_2 = 100)\) be the contract under consideration and let \(T = (w_1 = 401, w_2 = 99.7)\). Then \(EU(S) = 13.33, EV(S) = 16, EU(T) = 13.335\) and \(EV(T) = 16.004\), confirming that \(T\) is Pareto superior to \(S\).

**What if there are more than two outcomes?**

If there are more than two outcomes the same criterion applies. For example, with three outcomes \(x_1, x_2\) and \(x_3\) the necessary conditions for Pareto efficiency are

\[
\frac{U'(x_1 - w_1)}{U'(x_2 - w_2)} = \frac{V'(w_1)}{V'(w_2)} \quad \text{and} \quad \frac{U'(x_1 - w_1)}{U'(x_3 - w_3)} = \frac{V'(w_1)}{V'(w_3)}.
\]

Note that by dividing one by the other we get

\[
\frac{U'(x_2 - w_2)}{U'(x_3 - w_3)} = \frac{V'(w_2)}{V'(w_3)}
\]

so that this condition is implied by the other two.