MONEY LOTTERIES

Measuring risk aversion

How to identify risk aversion: $U''(x) < 0$

$\sqrt{x}$

$\ln(x)$ $\ln(x+1)$ etc.

Can there be more or less risk aversion?

Even the same utility function, **the degree of risk aversion of an individual varies with her level of wealth.**

$$\begin{pmatrix} -50 & 50 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} W_0-50 & W_0+50 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = L$$

$E[L] = W_0$

$U(x) = \sqrt{x}$. Initial wealth: W_0.

What is the **risk premium** associated with this lottery? **It depends on** W_0**.**

Suppose that $W_0 = 50$

$$L_1 = \begin{pmatrix} 0 & 100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$E[U(L_1)] = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{100} = \frac{1}{2}0 + \frac{1}{2}10 = 5$$

Solution to $\sqrt{\underbrace{50}_{E[L_1]} - R_L} = \underbrace{5}_{E[U(L)]}$

$R_L = \$25$

Suppose that Suppose that $W_0 = 1{,}000$

$$L_2 = \begin{pmatrix} 950 & 1{,}050 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

risk premium:

Solution to

$$\sqrt{\underbrace{1{,}000}_{E[L_2]} - R_L} = \underbrace{31.613}_{E[U(L_2)]}$$

$$E[U(L_2)] = \frac{1}{2}\sqrt{950} + \frac{1}{2}\sqrt{1050}$$
$$= 31.613$$

$$R_L = \$0.625$$

Thus she is less risk averse when her wealth is \$1,000 than when her wealth is \$50.

We compared two related lotteries **given some fixed preferences (i.e. a fixed utility function).**

Now **fix a lottery *L* and consider different preferences (that is, different utility functions)**.

Take the risk premium of the lottery as a measure of the intensity of risk aversion.

Initial wealth: 50. Wealth lottery: $L = \begin{pmatrix} 0 & 100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\mathbb{E}[L] = 50$

change it to

$7\sqrt{x} + 8$

- $\boxed{U(x) = \sqrt{x}}$ then, as we saw before, the risk premium is the solution to

  $\sqrt{50 - R} = \underbrace{5}_{=\mathbb{E}[U(L)]}$ which is $\boxed{R = \$25}$

- If her utility function is $\boxed{U(x) = \ln(x+1)}$

$E[U(L)] = \frac{1}{2}\ln(1) + \frac{1}{2}\ln(101)$

$= 2.3076$

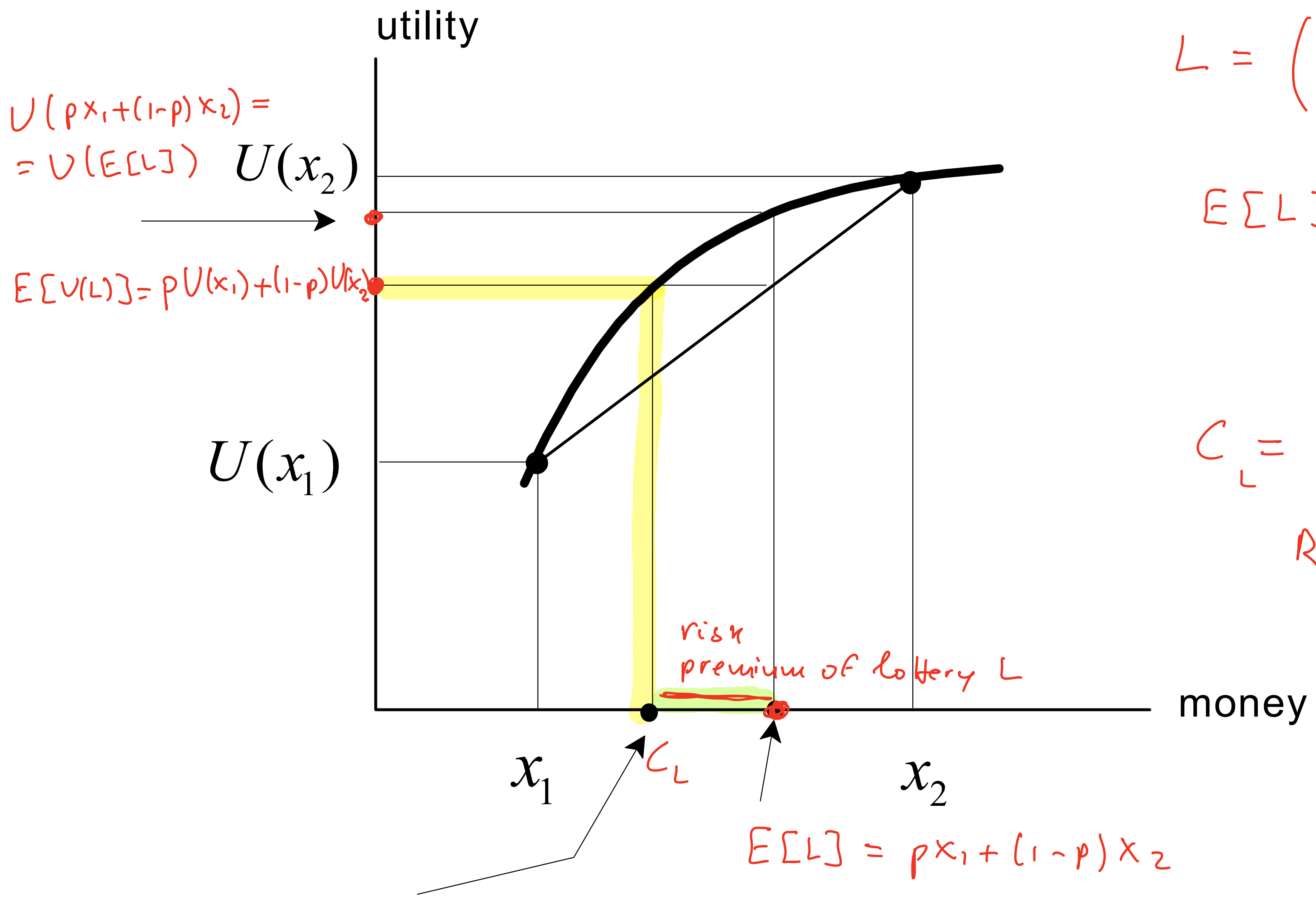
$\ln(50 - R_L + 1) = 2.3076$

$R_L = \$40.95$

Thus the utility function ln(x+1) embodies more risk aversion then the function $\sqrt{x}$ relative to lottery $\begin{pmatrix} 0 & 100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. But perhaps there is another lottery relative to which the function $\sqrt{x}$ displays more (or the same) risk aversion than the utility function ln(x +1)?

RISK AVERSE PERSON

Graphical representation of the risk premium:



$$L = \begin{pmatrix} x_1 & x_2 \\ p & 1-p \end{pmatrix} \quad 0<p<1$$

$$E[L] = p x_1 + (1-p) x_2$$

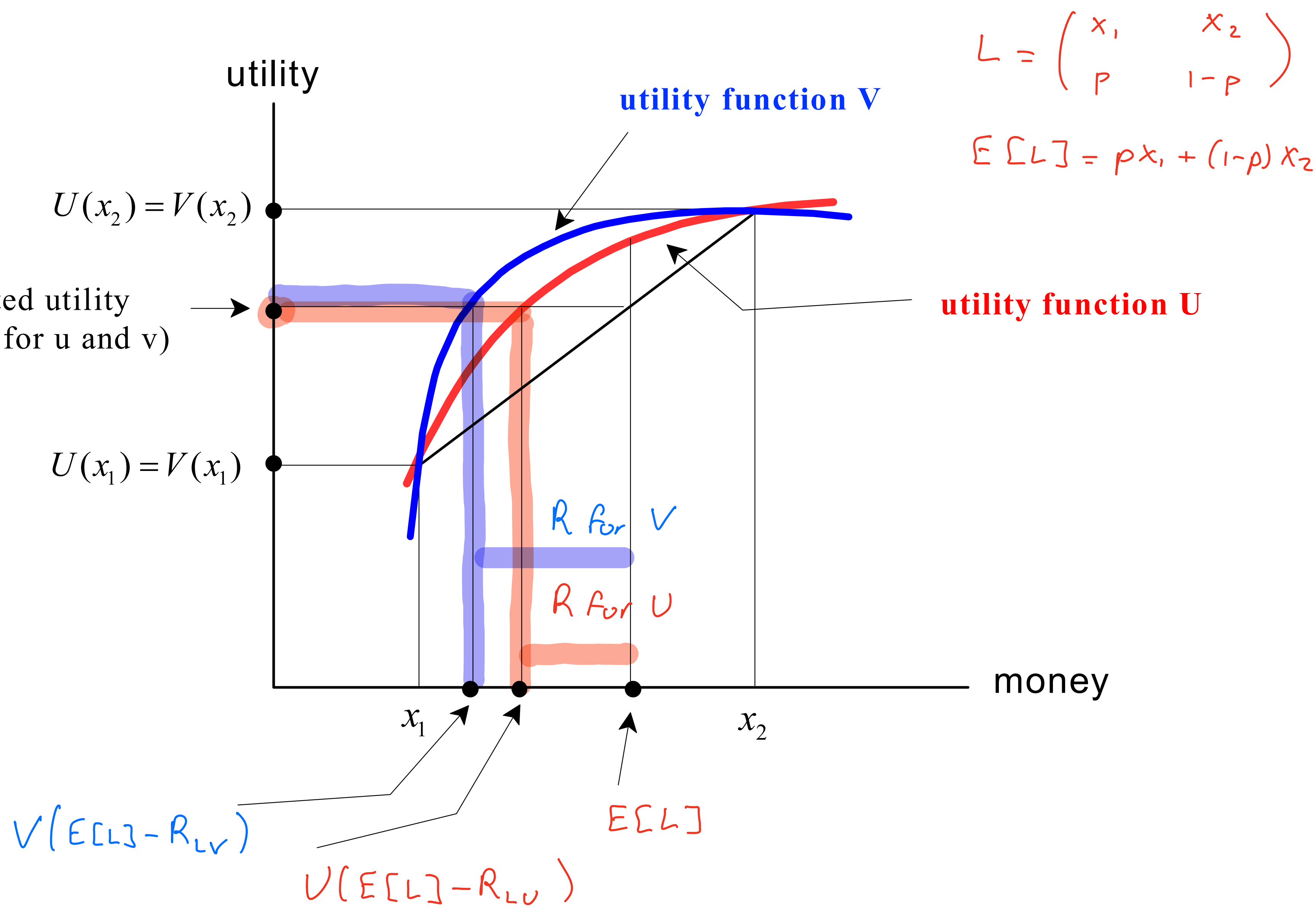
$$C_L = E[L] - R_L$$

$$R_L = E[L] - C_L$$

$$U(C_L) = E[U(L)]$$

certainty equivalent of lottery L

A more concave utility function is associated with a larger risk premium for the same lottery:



(2) Check that the risk premium is a meaningful measure, that is, that it is invariant to an allowed transformation of the utility function.

$$L = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}, \quad \mathbb{E}[L] = p_1 x_1 + p_2 x_2 + \dots + p_n x_n$$

Utility function $U(\$x)$. $\mathbb{E}[U(L)] =$ $p_1 U(x_1) + p_2 U(x_2) + \cdots + p_n U(x_n)$

R_{UL} solution to $\boxed{U(E[L] - R) = E[U(L)]}$

Now let $V(x) = aU(x) + b$ with $a > 0$

R_{VL} solution to $\underbrace{V(E[L]-R)}_{\parallel} = E[V(L)]$

$$\cancel{a}\, U(E[L]-R) + \cancel{b}$$

$$E[V(L)] = p_1 V(x_1) + p_2 V(x_2) \cdots + p_n V(x_n)$$

$$= p_1 (aU(x_1)+b) + p_2 (aU(x_2)+b) + \cdots + p_n (aU(x_n)+b)$$

$$= p_1 \cancel{a} U(x_1) + p_2 \cancel{a} U(x_2) + \cdots + p_n \cancel{a} U(x_n) + \cancel{b}\,\underbrace{(p_1 + p_2 + \cdots + p_n)}_{1}$$

$V(\mathbb{E}[L] - R) = \mathbb{E}[V(L)]$ if and only if $U(\mathbb{E}[L] - R) = \mathbb{E}[U(L)]$. Hence $\underline{R_{VL} = R_{UL}}$

Definition. Utility function U embodies more risk aversion that utility function V if

$R_{UL} > R_{VL}$ for every non-degenerate money lottery L.

Short of trying every possible lottery, is there a way to determine if U embodies more risk aversion than V?

Arrow-Pratt measure of risk aversion:

$$A = -\frac{\overbrace{U''(x)}^{-}}{\underbrace{U'(x)}_{+}} > 0$$

First, let us verify that it is a meaningful measure, that is, that it is invariant to an allowed transformation of the utility function

Let $\underbrace{V(x) = aU(x) + b}$ for every $x \geq 0$ with $\underbrace{a > 0}$. $V'(x) =$ and $V''(x) =$

$V'(x) = a\,U'(x)$

$V''(x) = a\,U''(x)$

$$\underbrace{-\frac{V''(x)}{V'(x)}}_{A_V} = -\frac{\cancel{a}\,U''(x)}{\cancel{a}\,U'(x)} = \underbrace{-\frac{U''(x)}{U'(x)}}_{A_U}$$

Examples.

$\boxed{U(x)=\sqrt{x}}=x^{\frac{1}{2}}$

$U'(x) = \frac{1}{2\sqrt{x}}$

$U''(x) = -\frac{1}{4\sqrt{x^3}}$

$A_{\sqrt{x}} = -\frac{-\frac{1}{4\sqrt{x^3}}}{\frac{1}{2}\sqrt{x}} = \frac{1}{2x}$

$\boxed{U(x)=\ln(x)}$

$U'(x) = \frac{1}{x}$

$U''(x) = -\frac{1}{x^2}$

$A_{\ln(\cdot)} = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x}$

$<$

Note that both display decreasing risk aversion as x increases

Theorem. Let U(x) and V(x) be two strictly concave functions. Then the following conditions are equivalent:

1. $R_{VL} > R_{UL}$ for every non-degenerate wealth lottery L

2. $\underbrace{A_V(x)}_{-\frac{V''(x)}{V'(x)}} > \underbrace{A_U(x)}_{-\frac{U''(x)}{U'(x)}}$ for every $x > 0$.