Ranking lotteries

Given two money lotteries L and M when would any two individuals agree that L is better than M, no matter their attitude to risk? Assume throughout that every individual prefers more money to less, that is, that each individual's utility function is strictly increasing.

Everybody will agree that $L = \begin{pmatrix} \$40 & \$120 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is better than $M = \begin{pmatrix} \$20 & \$100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

What about $L = \begin{pmatrix} \$0 & \$121 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $M = \begin{pmatrix} \$25 & \$100 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$?

$\mathbb{E}[L] = \frac{1}{2} 121 = 60.5$ and $\mathbb{E}[M] = \frac{3}{5} 25 + \frac{2}{5} 100 = 55$

For a risk-neutral person: $L > M$ if you are risk NEUTRAL

For a risk-averse person with utility function $U(x) = \sqrt{x}$

$\mathbb{E}[U(L)] = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{121} = \frac{1}{2} 11 = 5.5$ $\mathbb{E}[U(M)] = \frac{3}{5}\sqrt{25} + \frac{2}{5}\sqrt{100} = \frac{3}{5} 5 + \frac{2}{5} 10 = 7$

$M > L$ if risk averse with $U(x) = \sqrt{x}$

However, there are lotteries that can be unambiguously ranked in the sense that everybody ranks them the same way.

$A = \begin{pmatrix} \$100 & \$20 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$ rewrite it as $\begin{pmatrix} \$20 & \$100 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$

$$L = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \qquad M = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}.$$

Note that the **basic outcomes are the same** in both lotteries and for this part assume that the **prizes are listed in increasing order**: $0 \le x_1 < x_2 < \ldots < x_n$.

Define the **cumulative distribution function** (cdf) for lottery L as follows:

$P_i = p_1 + \ldots + p_i$ for every $i = 1, \ldots, n$:

$$L = \begin{array}{r} \\ \\ P: \end{array} \begin{pmatrix} \$x_1 & \$x_2 & \$x_3 & \dots & \$x_n \\ p_1 & p_2 & p_3 & \dots & p_n \\ P_1 & P_1 + P_2 & P_1 + P_2 + P_3 & & 1 \end{pmatrix}$$

$\leftarrow$ probability distribution or density function

$P(x_i) = P_1 + P_2 + \dots + P_i$

P_i is the probability that $x \le x_i$.

$P_i = P(x_i)$

define the cumulative probability distribution for lottery M as follows: $Q_i = q_1 + \ldots + q_i$ for every $i = 1, \ldots, n$:

$$M = \begin{array}{r} \\ \\ Q: \end{array} \begin{pmatrix} \$x_1 & \$x_2 & \$x_3 & \ldots & \$x_n \\ q_1 & q_2 & q_3 & \ldots & q_n \\ q_1 & q_1+q_2 & q_1+q_2+q_3 & & 1 \end{pmatrix}$$

cdf is P   cdf is Q

Definition. We say that L **first-order stochastically dominates** M and write $L >_{FSD} M$ if $P_i \le Q_i$ for ever $i = 1, 2, \ldots, n$, with at least one strict inequality.

Example 1.

$L = \begin{pmatrix} \$40 \\ 1 \end{pmatrix}$ and $M = \begin{pmatrix} \$20 & \$60 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

$$L = \begin{pmatrix} \$20 & \$40 & \$60 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{lccc} P \text{ cdf} & 0 & 1 & 1 \end{array}$$

$$M = \begin{pmatrix} \$20 & \$40 & \$60 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{array}{lccc} Q \text{ cdf} & \frac{1}{2} & \frac{1}{2} & 1 \end{array}$$

don't have that $P_i \le Q_i$ for all i

and don't have that $Q_i \le P_i$ for all i

L and M cannot be compared in terms of FSD

Example 2.

$$L = \begin{pmatrix} \$20 & \$40 & \$50 & \$60 \\ \frac{1}{12} & \frac{3}{12} & \frac{6}{12} & \frac{2}{12} \end{pmatrix} \text{ and } M = \begin{pmatrix} \$20 & \$40 & \$50 & \$60 \\ \frac{1}{12} & \frac{4}{12} & \frac{5}{12} & \frac{2}{12} \end{pmatrix}.$$

cdf $\frac{1}{12}$ $\frac{4}{12}$ $\frac{10}{12}$ 1 $\frac{1}{12}$ $\frac{5}{12}$ $\frac{10}{12}$ 1 cdf

L dominates M in the sense of FSD

$$L >_{FSD} M$$

Theorem. $L >_{FSD} M$ if and only if $\mathbb{E}[U(L)] > \mathbb{E}[U(M)]$ for every strictly increasing utility function U. So independently of attitude to risk

Thus if lottery L first-order stochastically dominates lottery M then it is unambiguously better than M, in the sense that everybody, no matter what their attitude to risk, prefers L to M.

Now **focus on risk-averse individuals** and ask when any two risk-averse individuals would agree that a lottery *M* is worse than another lottery *L*, in which case we can interpret this as ***M* being more risky than *L***.

To begin with the two lotteries ought to be similar: $\mathbb{E}[L] = \mathbb{E}[M]$, in which case a risk-neutral individual would be indifferent between the two. Hence if a risk-averse person is not indifferent it must be because one is "more risky" than the other.

$E[L] = 50$ $\quad L = \begin{pmatrix} \$50 \\ 1 \end{pmatrix} \succ M = \begin{pmatrix} \$0 & \$100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$E[M] = 50$

$$L = \begin{pmatrix} \$0 & \$50 & \$100 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\downarrow$$

$$M = \begin{pmatrix} \$0 & \$50 & \$100 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \quad \text{with} \quad \mathbb{E}[L] = 55$$

$$E[L] = \frac{1}{4}\,10 + \frac{1}{2}\,50 + \frac{1}{4}\,110$$

$\downarrow$ MEAN-PRESERVING SPREAD

$$M = \begin{pmatrix} \$10 & \$20 & \$50 & \$80 & \$110 \\ \frac{1}{4} & p & 0 & \frac{1}{2}-p & \frac{1}{4} \end{pmatrix} \quad \begin{matrix} \$50 \\ \frac{1}{2} \end{matrix}$$

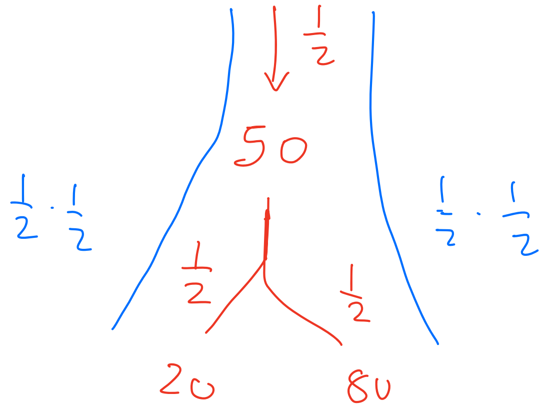
$$E[M] = \frac{1}{4}\,10 + p\,20 + \cancel{0\,50} + \left(\frac{1}{2}-p\right)80 + \frac{1}{4}\,110$$

$$\frac{1}{2}\,50 = 20p + \left(\frac{1}{2}-p\right)80$$

Solve for $p = \frac{1}{4}$

$$\begin{matrix} 20 & 50 & 80 \\ 0 & \frac{1}{2} & 0 \end{matrix}$$

$$\begin{matrix} 20 & 50 & 80 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{matrix}$$

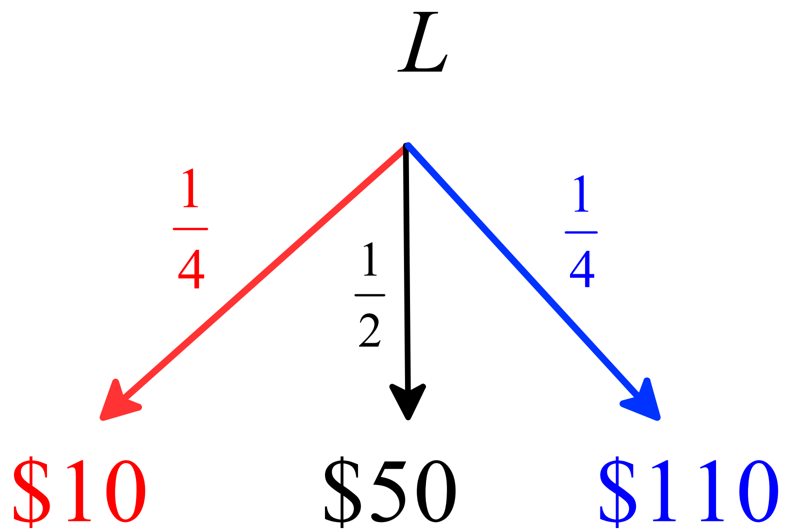


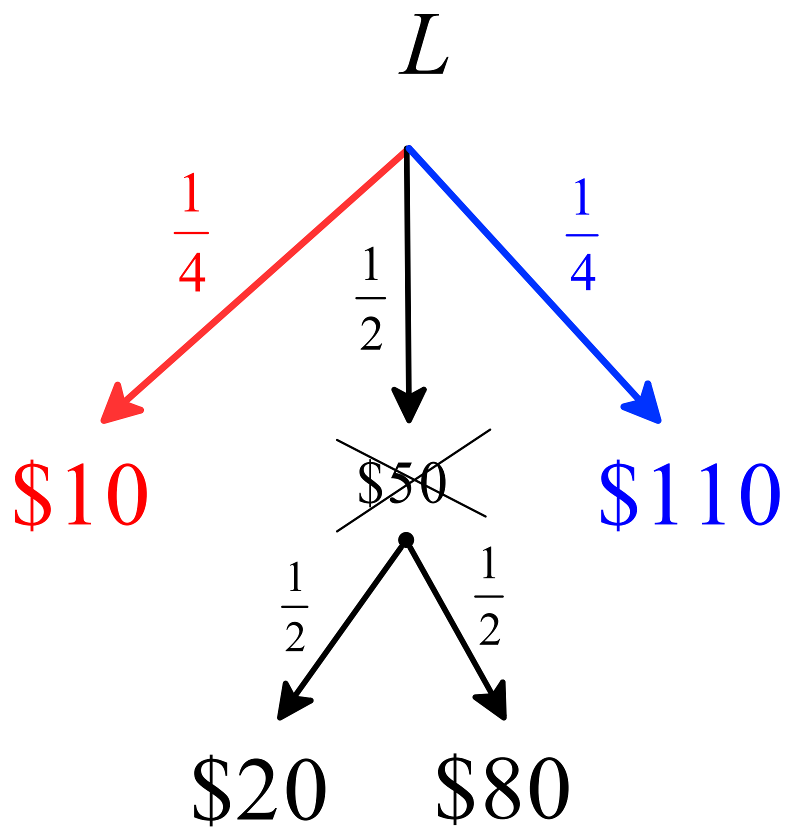
$$\begin{matrix} & 50 & \\ & \frac{1}{2} & \\ 20 & & 80 \\ \frac{1}{4} & & \frac{1}{4} \end{matrix}$$

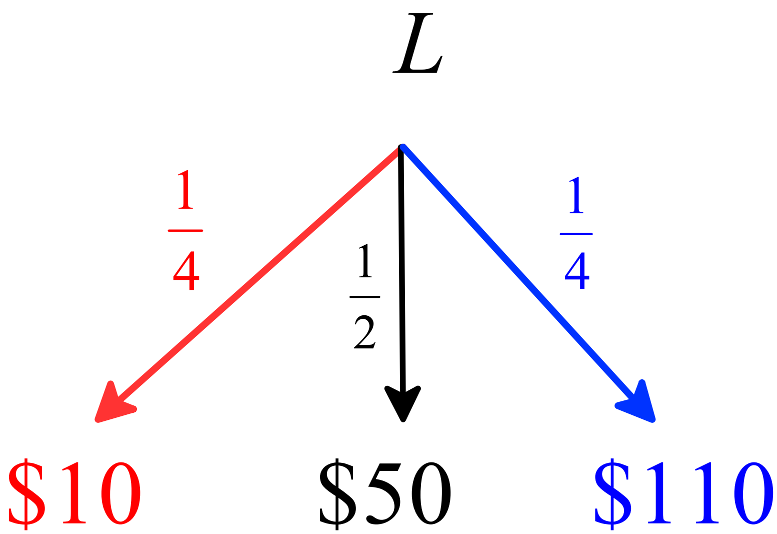
$$L = \begin{pmatrix} 10 & 20 & 50 & 80 & 110 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}$$

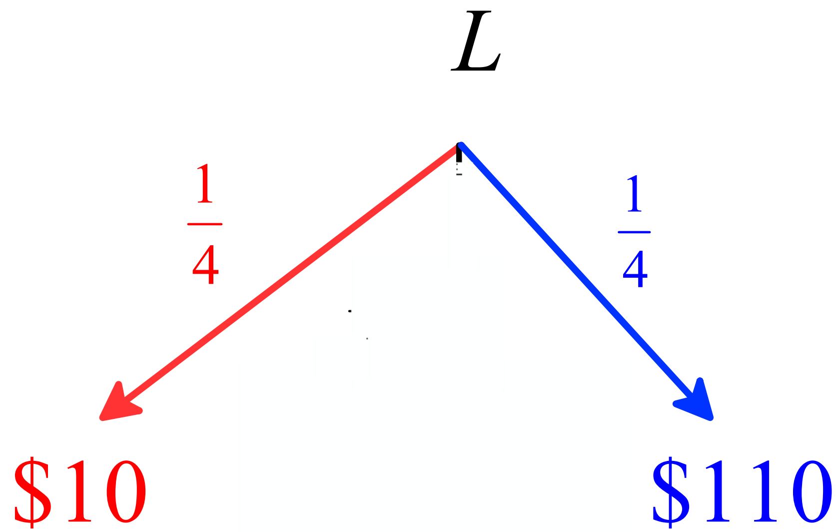
$$N = \begin{pmatrix} 10 & 20 & \begin{pmatrix} 20 & 80 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} & 80 & 110 \\ \frac{1}{4} & 0 & & 0 & \frac{1}{4} \end{pmatrix}$$

$$L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \text{ with } \quad \mathbb{E}[L] = 55$$









$$\begin{matrix} 10 & 50 & 110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{matrix} \rightarrow \begin{pmatrix} 10 & 20 & 50 & 80 & 110 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} \end{pmatrix}$$