We don't have to reduce the probability to zero:

$$L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} = \frac{5}{10} & \frac{1}{4} \end{pmatrix}$$

Take away some of the probability of \$50, say $\frac{3}{10}$ and spread it between a lower amount, say \$15, and a higher amount, say \$90:

$$M = \begin{pmatrix} \$10 & \$15 & \$50 & \$90 & \$110 \\ \frac{1}{4} & r & \frac{5}{10} - \frac{3}{10} = \frac{2}{10} & s & \frac{1}{4} \end{pmatrix}$$

For this to be a mean preserving spread we need

$$\boxed{\begin{aligned} &(1) \quad r + s = \frac{3}{10} \\ &(2) \quad \frac{3}{10} 50 = 15r + 90s \end{aligned}}$$

in computation of $E[M] = \frac{2}{10} 50 + r15 + s90$

in computation of $E[L] = \frac{5}{10} 50 = \frac{2}{10} 50 + \frac{3}{10} 50$

need these to be equal in order to have $E[L] = E[M]$

$$M = \begin{pmatrix} \$10 & \$15 & \$50 & \$90 & \$110 \\ \frac{1}{4} & \frac{4}{25} & \frac{2}{10} & \frac{7}{50} & \frac{1}{4} \end{pmatrix}$$

Solution is $r = \frac{4}{25}$, $s = \frac{7}{50}$

Write $L >_{SSD} M$ to mean that ***L* dominates *M*** in the sense of **second-order stochastic dominance.**

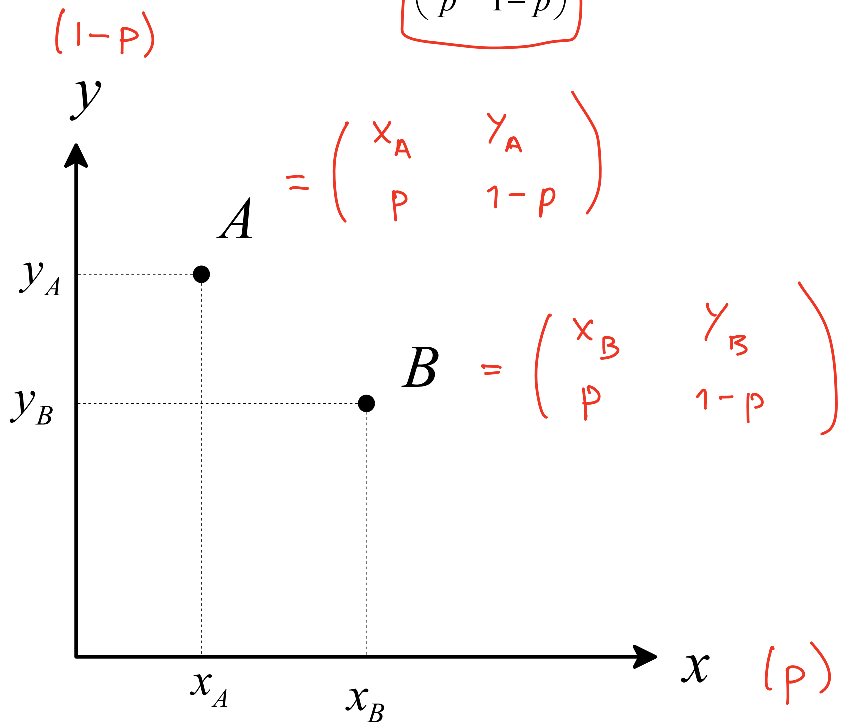
Definition. $L >_{SSD} M$ if M can be obtained from L by a finite sequence of mean - preserving spreads, that is, if there is a sequence of money lotteries $\langle L_1, L_2, \ldots, L_m \rangle$ (with $m \geq 2$) such that:

(1) $L_1 = L,$
(2) $L_m = M$
(3) for every $i = 1, \ldots, m-1,\ L_i \rightarrow_{MPS} L_{i+1}$

Theorem. $L >_{SSD} M$ if and only if $\mathbb{E}[U(L)] > \mathbb{E}[U(M)]$ for every strictly increasing and strictly concave utility function U.

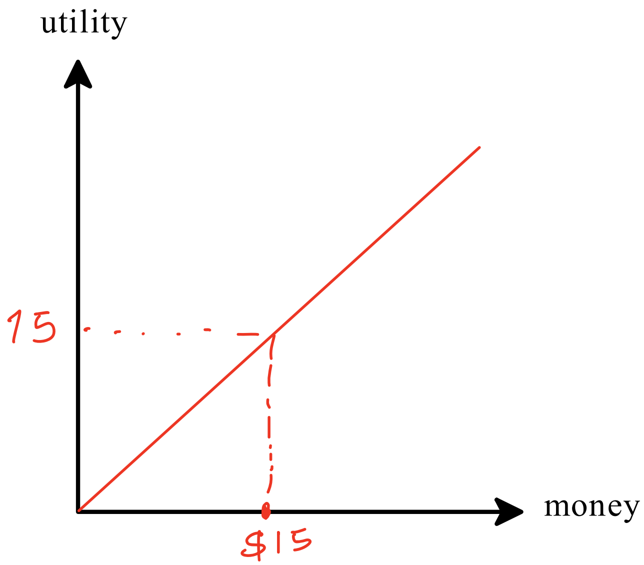
BINARY LOTTERIES

Lotteries of the form $\boxed{\begin{pmatrix} \$x & \$y \\ p & 1-p \end{pmatrix}}$ with p fixed and x and y allowed to vary.



We want to draw indifference curves in this diagram.

Case 1: risk-neutral agent



$U(x) = x$ identity function

more generally $V(x) = ax + b$ with $a > 0$

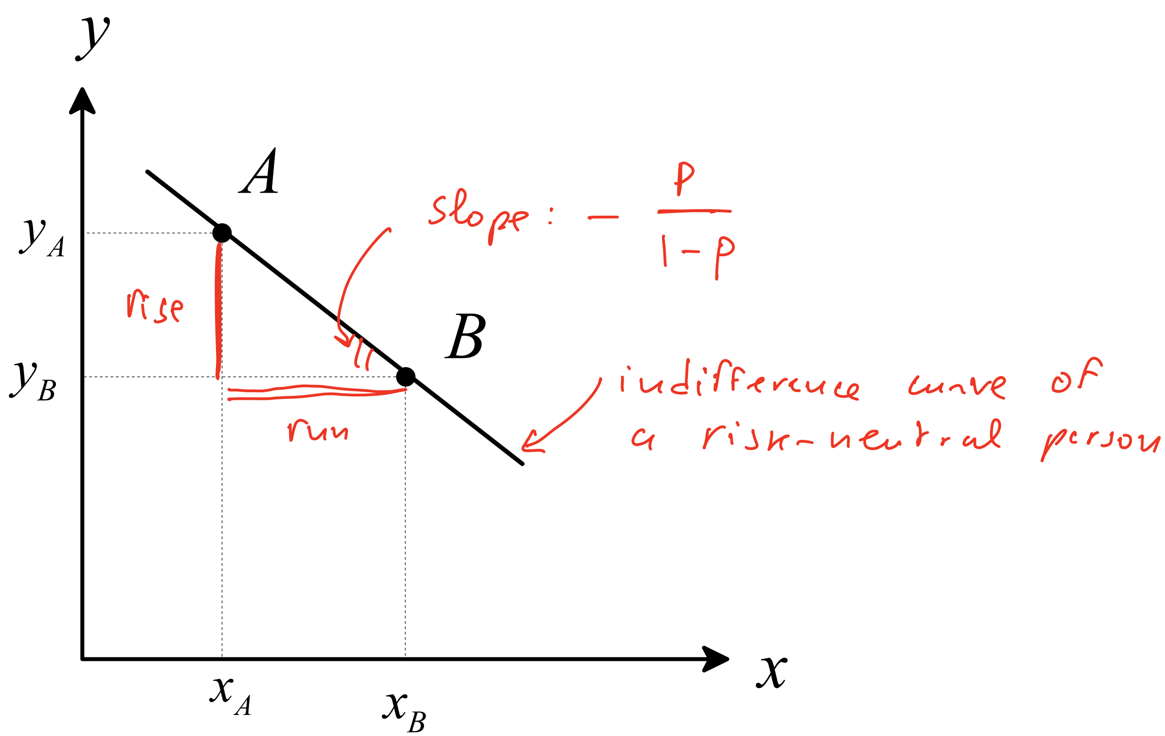
$$A = \begin{pmatrix} x_A & y_A \\ p & 1-p \end{pmatrix} \qquad B = \begin{pmatrix} x_B & y_B \\ p & 1-p \end{pmatrix}$$

Let A and B be such that $\underbrace{\mathbb{E}[U(A)]}_{=E[A]} = \underbrace{\mathbb{E}[U(B)]}_{=E[B]}$:

$$p x_A + (1-p) y_A = p x_B + (1-p) y_B$$

$$(1-p)(y_A - y_B) = -p(x_A - x_B)$$

$$\frac{\overbrace{y_A - y_B}^{\text{rise}}}{\underbrace{x_A - x_B}_{\text{run}}} = -\frac{p}{1-p}$$


y
x
A
B
y_A
y_B
x_A
x_B
rise
run
slope: $-\frac{p}{1-p}$
indifference curve of
a risk-neutral person

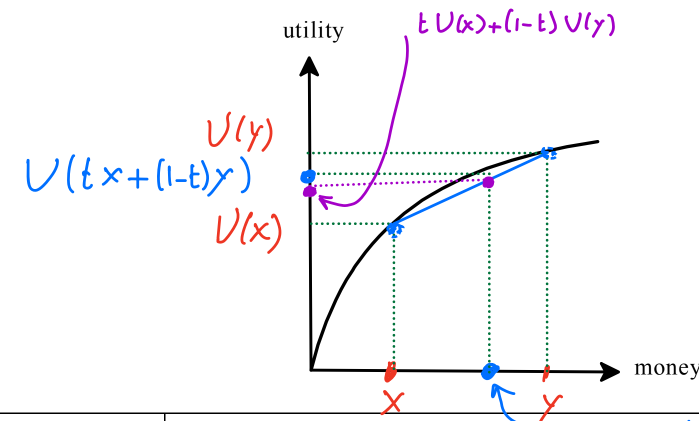
Case 2: risk-averse agent

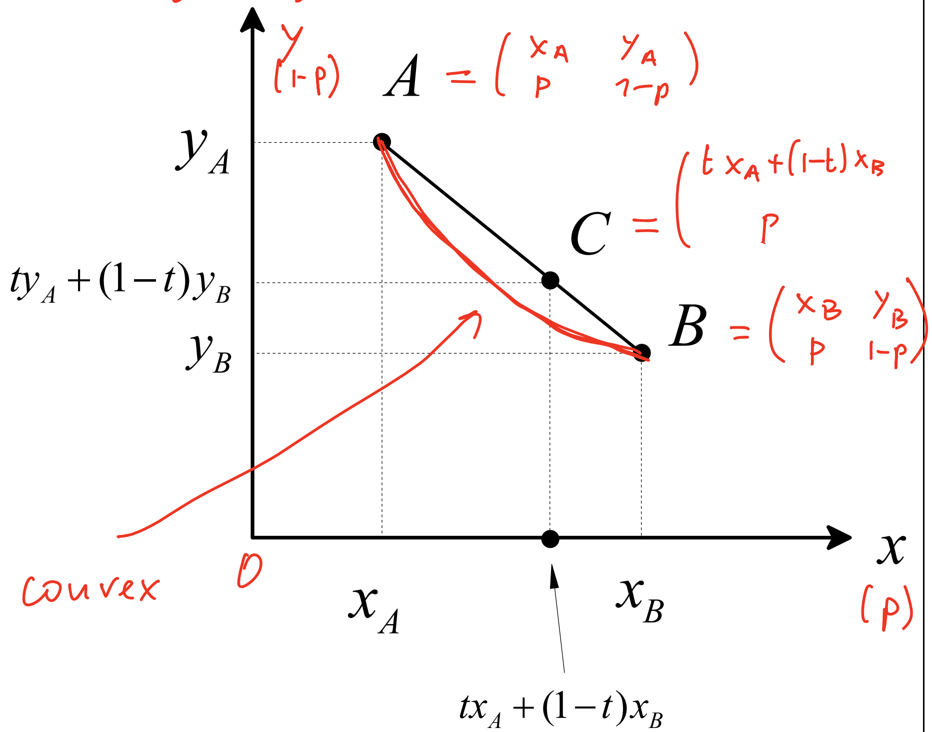
Fix a $0<t<1$

$tx+(1-t)y$

$U(m)$ is strictly concave: if for all m_1, m_2 (with $0 \le m_1 < m_2$) and all $t \in (0,1)$

$$\underbrace{U(tm_1+(1-t)m_2)}_{\text{utility of expected value of lottery } \begin{pmatrix} m_1 & m_2 \\ t & 1-t \end{pmatrix}} > \underbrace{tU(m_1)+(1-t)U(m_2)}_{\text{expected utility of lottery}}$$





$A = \begin{pmatrix} x_A & y_A \\ p & 1-p \end{pmatrix}$

$C = \begin{pmatrix} tx_A+(1-t)x_B & ty_A+(1-t)y_B \\ p & 1-p \end{pmatrix}$

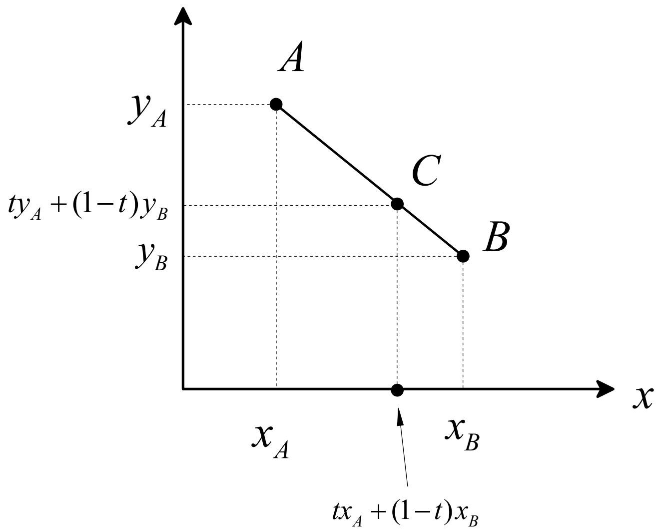
$B = \begin{pmatrix} x_B & y_B \\ p & 1-p \end{pmatrix}$

Suppose that $\mathbb{E}[U(A)] = \mathbb{E}[U(B)]$

Want to show that
$\mathbb{E}[U(C)] > \mathbb{E}[U(A)] (= \mathbb{E}[U(B)])$

$[tU(x_A)+(1-t)U(x_B)]$

by definition of concavity



$$E[U(C)] = p\,\overbrace{U(t x_A + (1-t) x_B)} + (1-p)\,\underbrace{U(t y_A + (1-t) y_B)}$$

$$\geq$$

$$t\,U(y_A) + (1-t)\,U(y_B)$$

$\mathbb{E}[U(C)] =$

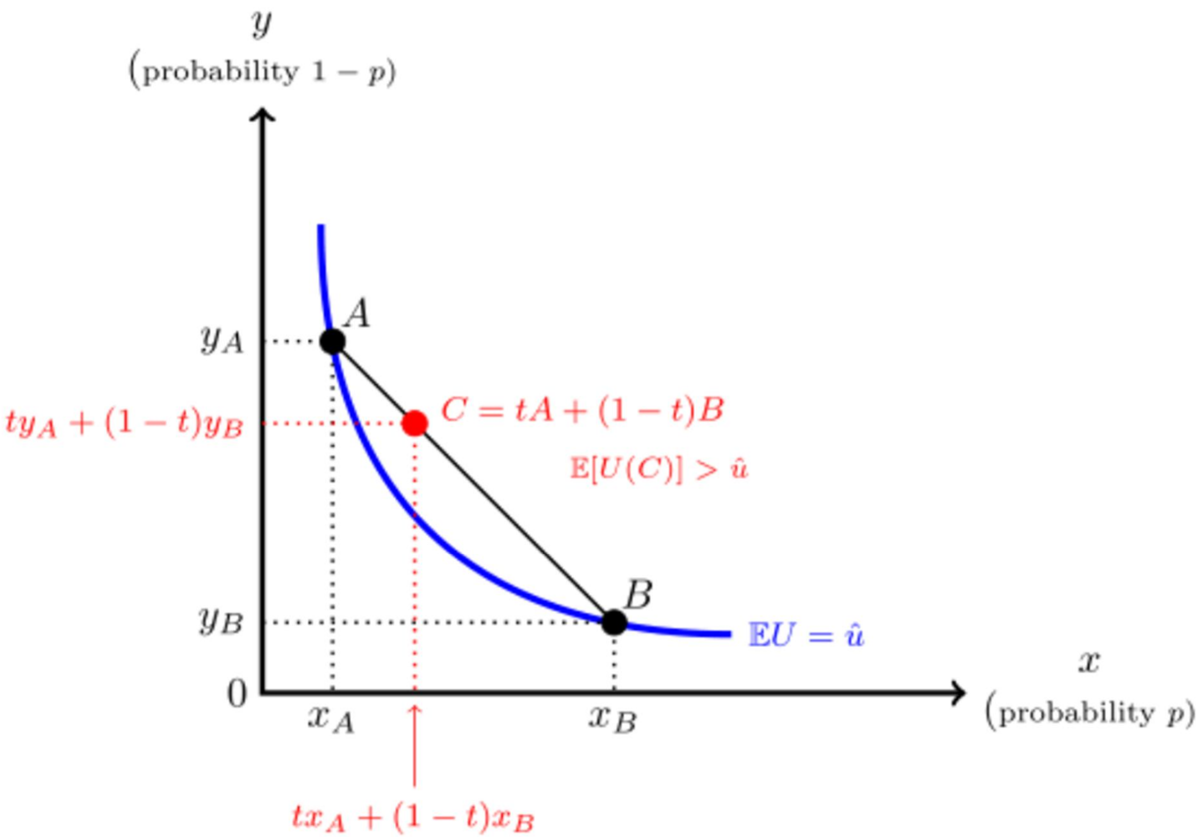
$$E[U(C)] > p\,[\underbrace{t\,U(x_A)} + (1-t)\,U(x_B)] + (1-p)\,[\underbrace{t\,U(y_A)} + (1-t)\,U(y_B)]$$

$$= \underbrace{p\,t\,U(x_A) + (1-p)\,t\,U(y_A)} + \underbrace{p\,(1-t)\,U(x_B) + (1-p)\,(1-t)\,U(y_B)}$$

$$t\,\underbrace{[p\,U(x_A) + (1-p)\,U(y_A)]}_{E[U(A)]} + (1-t)\cdot\underbrace{[p\,U(x_B) + (1-p)\,U(y_B)]}_{E[U(B)] \,=\, E[U(A)]}$$

$$= E[U(A)]$$

The indifference curve must lie below the straight-line segment joining A and B.


y
(probability $1-p$)
A
y_A
$ty_A + (1-t)y_B$
$C = tA + (1-t)B$
$\mathbb{E}[U(C)] > \hat{u}$
B
y_B
$\mathbb{E}U = \hat{u}$
x
(probability p)
0
x_A
x_B
$tx_A + (1-t)x_B$

Case 2: risk-loving agent