New topic: Asymmetric information and Adverse Selection

Probability and conditional probability

Finite set of ***states*** $S=\{s_1,s_2,\ldots,s_n\}$. Subsets of S are called ***events***.

Probability distribution over S:

For every $i=1,\ldots,n$

$0\le p_i\le 1$

$\sum_{j=1}^{n} p_j = 1$

$$\begin{array}{cccc} s_1 & s_2 & \cdots & s_n \\ p_1 & p_2 & \cdots & p_n \end{array}$$

Denote the probability of state s by $p(s)$.

Given an event $E\subseteq S$, the probability of E is:

$$P(E)=\begin{cases} 0 & \text{if } E=\emptyset \\ \sum_{s\in E} p(s) & \text{if } E\neq\emptyset \end{cases}$$

$\overline{E}$

Denote by $\neg E$ the complement of $E\subseteq S$.

all the states in S that are not in E

Example

$S = \{a, b, c, d, e, f, g\}$ $A = \{a, c, d, e\}$ $B = \{a, e, g\}$

$\neg A = \{b, f, g\}$ $\neg B = \{b, c, d, f\}$ $P(\neg E) = 1 - P(E)$

Given

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$P(A) = \frac{1}{14} + 0 + \frac{1}{14} + \frac{6}{14} = \frac{8}{14}$ $P(B) = \frac{1}{14} + \frac{6}{14} + \frac{3}{14} = \frac{10}{14}$

$A \cap B = \{a, e\}$ $P(A \cap B) = \frac{1}{14} + \frac{6}{14} = \frac{7}{14}$

$A \cup B = \{a, c, d, e, g\}$ $P(A \cup B) = \frac{1}{14} + 0 + \frac{1}{14} + \frac{6}{14} + \frac{3}{14} = \frac{11}{14}$

Note: for every two events E and F: $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{8}{14} + \frac{10}{14} - \frac{7}{14} = \frac{11}{14}$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$P(A | B) = P(\{a, c, d, e\} | \{a, e, g\}) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{7}{14}}{\frac{10}{14}} = \frac{7}{10}$$

We denote by $P(E\mid F)$ the probability of E **conditional on** F and define it as:

$$P(E\mid F)=\frac{P(E\cap F)}{P(F)} \qquad \text{assuming } P(F)\neq 0$$

Probability of E conditional on (or given) F

Continuing the example above where

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$A=\{a,c,d,e\}$ $\quad B=\{a,e,g\}$

$P(A)=\frac{1}{14}+0+\frac{1}{14}+\frac{6}{14}=\frac{8}{14}$, $\quad P(B)=\frac{1}{14}+\frac{6}{14}+\frac{3}{14}=\frac{10}{14}$, $\quad P(A\cap B)=\frac{1}{14}+\frac{6}{14}=\frac{7}{14}$

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{7}{10}$$

$$P(B\mid A)=\frac{P(B\cap A)}{P(A)}=\frac{P(A\cap B)}{P(A)}=\frac{\frac{7}{14}}{\frac{8}{14}}=\frac{7}{8}$$

The conditional probability formula can also be applied to individual states:

$F=\{s\}$

$P(F\mid E)=p(s\mid E)$

$$p(s\mid E)=\begin{cases}\dfrac{P(\emptyset)}{P(E)}=\dfrac{0}{P(E)}=0 & \text{if } s\notin E\\[2ex] \dfrac{p(s)}{P(E)} & \text{if } s\in E\end{cases} \qquad \text{assuming } P(E)\neq 0$$

We can think of $p(\bullet|E)$ as a probability distribution on the entire set S. Continuing the example above

where $S = \{a,b,c,d,e,f,g\}$, $A = \{a,c,d,e\}$ and $\begin{pmatrix} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{pmatrix}$ P (so that $P(A) = \frac{8}{14}$)

$$\frac{P(a)}{P(A)} = \frac{\frac{1}{14}}{\frac{8}{14}} = \frac{1}{8}$$

$p(\bullet|A)$: $\begin{pmatrix} a & b & c & d & e & f & g \\ \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{6}{8} & 0 & 0 \end{pmatrix}$

$1+0+1+6=8$

Shortcut to obtain the revised or updated probabilities:

Initial or prior probabilities. Note that here they all have the **same denominator**.	$\begin{pmatrix} a & b & c & d \\ \frac{15}{100} & \frac{70}{100} & \frac{5}{100} & \frac{10}{100} \end{pmatrix}$
Information or conditioning event: $F = \{a,b,d\}$	
STEP 1. Set the probability of every state which is not in F to zero:	$\begin{pmatrix} a & b & c & d \\ & & 0 & \end{pmatrix}$
STEP 2. For the other states write new fractions with the same numerators as before:	$\begin{pmatrix} a & b & c & d \\ \frac{15}{\dots} & \frac{70}{\dots} & 0 & \frac{10}{\dots} \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 15+70+10=95. Thus the updated probabilities are:	$\begin{pmatrix} a & b & c & d \\ \frac{15}{95} & \frac{70}{95} & 0 & \frac{10}{95} \end{pmatrix}$

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$15+70+10=95$

EXAMPLE 2. Sample space or set of states: $\{a, b, c, d, e, f\}$.

$\frac{3}{20} \quad \frac{6}{20} \quad \frac{1}{20} \quad 0 \quad \frac{8}{20} \quad \frac{2}{20} \leftarrow$

Initial or prior probabilities:	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{20} & \frac{3}{10} & \frac{1}{20} & 0 & \frac{2}{5} & \frac{1}{10} \end{pmatrix}$
Information:	$F = \{a, b, d, e\}$
STEP 0. Rewrite all the probabilities **with the same denominator:**	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \end{pmatrix}$
STEP 1. Change the probability of every state which is not in F to zero:	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \end{pmatrix}$
STEP 2. Write new fractions which have the same numerators as before:	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{\ } & \frac{6}{\ } & 0 & \frac{0}{\ } & \frac{8}{\ } & 0 \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 3+6+8=17. $p(\cdot \mid F)$	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{17} & \frac{6}{17} & 0 & 0 & \frac{8}{17} & 0 \end{pmatrix}$

$3 + 6 + 0 + 8 = 17$

ADVERSE SELECTION
Akerlof on market for second-hand cars

Utility-of-money of a potential seller who owns of a car of quality q:

$$U(m) = \begin{cases} m + u(q) & \text{if does not sell the car} \\ m & \text{if sells the car} \end{cases}$$

$u(q)$ = value of a car of quality q to the owner of such a car

Thus, if her initial wealth is W_0 she will sell the car a price p only if:

before sale utility : $W_0 + u(q)$

after sale at price p " : $W_0 + p$

willing to sell if $W_0 + p \geq W_0 + u(q)$ i.e. if $\underline{p \geq u(q)}$

Utility-of-money of a potential buyer who does not own a car:

$$V(m) = \begin{cases} \text{m} & \text{if does not buy a car} \\ m + v(q) & \text{if becomes owner of a car of quality } q \end{cases}$$

$v(q)$ = value of car of quality q to the potential buyer

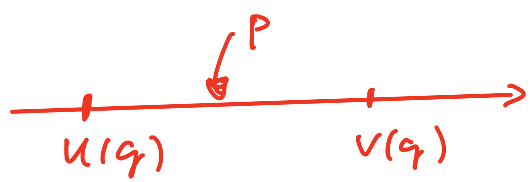
Thus, if his initial wealth is W_0 he will but a car of quality q at price p only if:

before purchase utility : W_0

after purchase at price p " : $W_0 - p + v(q)$

willing to buy if $W_0 - p + v(q) \geq W_0$ i.e. if $\underline{v(q) \geq p}$

Assume that, for every quality q, $\underline{v(q) > u(q) > 0}$



Both buyers and sellers are risk neutral

$v(q) > u(q)$

What if there is **asymmetric information**: only the owner knows the quality q?

Publicly available information:

Quality q	best: A	B	C	D	E	worst: F	
Number of cars	120	200	100	240	320	140	Total: 1,120
Proportion	$\frac{120}{1120}$	$\frac{200}{1120}$	$\frac{100}{1120}$	$\frac{240}{1120}$	$\frac{320}{1120}$	$\frac{140}{1120}$	
$v(q)$ (seller)	720	630	540	450	360	270	
$u(q)$ (buyer)	800	700	600	500	400	300	

Buyer: if a car is offered to me at price p should I buy it?

getting a car
$$\begin{pmatrix} \$800 & \$700 & \$600 & \$500 & \$400 & \$300 \\ \frac{120}{1120} & \frac{200}{1120} & \frac{100}{1120} & \frac{240}{1120} & \frac{320}{1120} & \frac{140}{1120} \end{pmatrix}$$