Probability and conditional probability

Finite set of ***states*** $S = \{s_1, s_2, ..., s_n\}$. Subsets of S are called ***events***.

Probability distribution over *S:*

$$\begin{array}{cccc} s_1 & s_2 & \cdots & s_n \\ p_1 & p_2 & \cdots & p_n \end{array}$$

Denote the probability of state s by $p(s)$.

Given an event $E \subseteq S$, the probability of E is:

$$P(E) = \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

Denote by $\neg E$ the complement of $E \subseteq S$.

Example

$S = \{a, b, c, d, e, f, g\}$ $\quad A = \{a, c, d, e\}$ $\quad B = \{a, e, g\}$

$\neg A =$ $\quad \neg B =$

Given

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$P(A) =$ $\quad P(B) =$

$A \cap B =$ $\quad P(A \cap B) =$

$A \cup B =$ $\quad P(A \cup B) =$.

Note: for every two events E and F:

$$P(E \cup F) =$$

We denote by $P(E \mid F)$ the probability of E **conditional on** F and define it as:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Continuing the example above where

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$A = \{a, c, d, e\}$ $\qquad B = \{a, e, g\}$

$P(A) = \frac{1}{14} + 0 + \frac{1}{14} + \frac{6}{14} = \frac{8}{14}$, $\quad P(B) = \frac{1}{14} + \frac{6}{14} + \frac{3}{14} = \frac{10}{14}$, $\quad P(A \cap B) = \frac{1}{14} + \frac{6}{14} = \frac{7}{14}$

$$P(A \mid B) =$$

$$P(B \mid A) = \qquad .$$

The conditional probability formula can also be applied to individual states:

$$p(s \mid E) = \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

We can think of $p(\bullet|E)$ as a probability distribution on the entire set *S*. Continuing the example above where $S=\{a,b,c,d,e,f,g\}$, $A=\{a,c,d,e\}$ and $\begin{matrix} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{matrix}$ (so that $P(A)=\frac{8}{14}$)

$$\begin{matrix} & a & b & c & d & e & f & g \\ p(\bullet|A): & & & & & & & \end{matrix}$$

Shortcut to obtain the revised or updated probabilities:

Initial or prior probabilities. Note that here they all have the **same denominator**.	$\begin{pmatrix} a & b & c & d \\ \frac{15}{100} & \frac{70}{100} & \frac{5}{100} & \frac{10}{100} \end{pmatrix}$
Information or conditioning event: $F=\{a,b,d\}$	
STEP 1. Set the probability of every state which is not in *F* to zero:	$\begin{pmatrix} a & b & c & d \\ & & 0 & \end{pmatrix}$
STEP 2. For the other states write new fractions with the same numerators as before:	$\begin{pmatrix} a & b & c & d \\ \frac{15}{\ldots} & \frac{70}{\ldots} & 0 & \frac{10}{\ldots} \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 15+70+10=95. Thus the updated probabilities are:	$\begin{pmatrix} a & b & c & d \\ \frac{15}{95} & \frac{70}{95} & 0 & \frac{10}{95} \end{pmatrix}$

In the above example, where

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

and $A = \{a, c, d, e\}$, to compute $p(\bullet | A)$

Step 1: assign zero probability to states in $\neg A$:

a	b	c	d	e	f	g

Step 2: keep the same numerators for the states in A:

a	b	c	d	e	f	g
	0				0	0

Step 3: since the sum of the numerators is 8, put 8 as the denominator:

a	b	c	d	e	f	g
$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\frac{6}{8}$	0	0

EXAMPLE 2. Sample space or set of states: $\{a, b, c, d, e, f\}$.

Initial or prior probabilities:	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{20} & \frac{3}{10} & \frac{1}{20} & 0 & \frac{2}{5} & \frac{1}{10} \end{pmatrix}$
Information:	$F = \{a, b, d, e\}$
STEP 0. Rewrite all the probabilities **with the same denominator:**	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \end{pmatrix}$
STEP 1. Change the probability of every state which is not in F to zero:	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \end{pmatrix}$
STEP 2. Write new fractions which have the same numerators as before:	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 3+6+8=17.	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{17} & \frac{6}{17} & 0 & 0 & \frac{8}{17} & 0 \end{pmatrix}$

ADVERSE SELECTION
Akerlof on market for second-hand cars

Utility-of-money of a potential seller who owns of a car of quality q:

$$U(m) = \begin{cases} m + u(q) & \text{if does not sell the car} \\ m & \text{if sells the car} \end{cases}$$

Thus, if her initial wealth is W_0 she will sell the car a price p only if:

Utility-of-money of a potential buyer who does not own a car:

$$V(m) = \begin{cases} \text{m} & \text{if does not buy a car} \\ m + v(q) & \text{if becomes owner of a car of quality } q \end{cases}$$

Thus, if his initial wealth is W_0 he will but a car of quality q at price p only if:

Assume that, for every quality q, $v(q) > u(q) > 0$

What if there is **asymmetric information**: only the owner knows the quality q?

Publicly available information:

Quality q	best: A	B	C	D	E	worst: F	
Number of cars	120	200	100	240	320	140	Total: 1,120
Proportion							
$v(q)$ (seller)	720	630	540	450	360	270	
$u(q)$ (buyer)	800	700	600	500	400	300	

Buyer: if a car is offered to me at price p should I buy it?

Suppose $p = 460$

Quality q	best: A	B	C	D	E	worst: F
$v(q)$ (seller)	720	630	540	450	360	270

Back to previous example. Suppose that p = 460. Then only qualities D, E, F offered

Step 1: convert probabilities to a common denominator:

Quality q	best: A	B	C	D	E	worst: F
Proportion	$p_A = \frac{3}{28}$	$p_B = \frac{5}{28}$	$p_C = \frac{5}{56}$	$p_D = \frac{3}{14}$	$p_E = \frac{2}{7}$	$p_F = \frac{1}{8}$

Step 2: condition on {D, E, F}

Quality q	best: A	B	C	D	E	worst: F
Proportion						

Suppose $p = 380$

Quality q	best: A	B	C	D	E	worst: F
$v(q)$ (seller)	720	630	540	450	360	270

Quality	L	M	H
probability	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$
seller's value	900	$1,200$	$1,400$
buyer's value	$1,020$	$1,320$	$1,500$

For every price p determine if there is a second-hand market.

All-you-can eat buffet in Davis. Hire a market research firm to find out about demand. Customers of different types. A type of a customer is a pair (r, c) where

- r is the maximum price the customer is willing to pay
- c is the number of dishes that the customer would consume

Customer type	(\$8, 2)	(\$8, 2.5)	(\$8.50, 2.5)	(\$8.50, 3)	(\$9, 3)	(\$9, 3.5)
Proportion	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{7}{24}$

Risk neutral. **Cost per dish is \$2.40**.

- If you charge \$8 then average consumption

  Average cost per customer

  Profit per customer

What if you charge $8.50?

Customer type	($8, 2)	($8, 2.5)	($8.50, 2.5)	($8.50, 3)	($9, 3)	($9, 3.5)
Proportion	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{7}{24}$

Step 1: convert to same denominator

Customer type	($8, 2)	($8, 2.5)	($8.50, 2.5)	($8.50, 3)	($9, 3)	($9, 3.5)
Proportion						

- If you charge $8.50 then

Customer type	($8.50, 2.5)	($8.50, 3)	($9, 3)	($9, 3.5)
Proportion				

Average consumption:

Average cost per customer

Profit per customer

- If you charge $9 then

Customer type	($9, 3)	($9, 3.5)
Proportion		

Average consumption:

Average cost per customer

Profit per customer