People tend to be risk-averse towards gains, but risk-loving towards losses.

Can such an attitude be compatible with expected utility?

Choice between $A:\begin{pmatrix} +\$50 \\ 1 \end{pmatrix}$ and $B:\begin{pmatrix} +\$100 & +\$0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $A \succ B$

Suppose that **she prefers** the sure gain: she prefers **A**. Then she displays **risk-aversion towards gains** (the expected value of these two options is the same). .

Choice between $C:\begin{pmatrix} -\$50 \\ 1 \end{pmatrix}$ and $D:\begin{pmatrix} -\$100 & -\$0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. $D \succ C$

Suppose that **she prefers** the risky prospect: she prefers **D**. **Then she is risk-loving towards losses** (the expected value of these two options is the same).

Is there a von Neumann-Morgenstern utility function that is consistent with these choices?

Suppose that her initial wealth is $100.

outcome	U (normalized)
$200	1
$150	a
$100	b
$50	c
$0	0

$1 > a > b > c > 0$

$$A = \begin{pmatrix} +\$50 \\ 1 \end{pmatrix} \succ B = \begin{pmatrix} +\$100 & +\$0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$EU(A) =$
$U(150) = a$

$EU(B) = \frac{1}{2} U(200) + \frac{1}{2} U(100)$
$= \frac{1}{2} 1 + \frac{1}{2} b$

From $A \succ B$ infer $a > \frac{1}{2} + \frac{1}{2} b$

$\boxed{2a > 1 + b}$ ✓

Hence it is possible for an expected-utility maximizing individual to display risk aversion towards a gain and risk love towards a symmetric loss.

$a = 0.8$ $2 \cdot a = 1.6$
$b = 0.5$ $1 + b = 1.5$

outcome	normalized
\$200	1
\$150	a
\$100	b
\$50	c
\$0	0

$$C: \begin{pmatrix} -\$50 \\ 1 \end{pmatrix} \quad \text{and} \quad D: \begin{pmatrix} -\$100 & -\$0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$$D \succ C$$

$$\underbrace{EV(D)}_{} > \underbrace{EV(C)}_{= U(50) = c}$$

$$\frac{1}{2}U(0) + \frac{1}{2}U(100)$$

$$= \frac{1}{2} b$$

From $D \succ C$ infer $\frac{1}{2} b > c$

or $\boxed{b > 2c}$ ✓

$c = 0.2$

$0.5 > 0.4$

Yes it is possible: for example

$a = 0.8$
$b = 0.5$
$c = 0.2$

However, this cannot happen at every wealth level.

Beginning wealth: \$200. Choice between $A:\begin{pmatrix} +\$50 \\ 1 \end{pmatrix}$ and $B:\begin{pmatrix} +\$100 & +\$0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

$A = \begin{pmatrix} 250 \\ 1 \end{pmatrix}$ $B = \begin{pmatrix} 300 & 200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Beginning wealth: \$200. Choice between $C:\begin{pmatrix} -\$50 \\ 1 \end{pmatrix}$ and $D:\begin{pmatrix} -\$100 & -\$0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

$C = \begin{pmatrix} 150 \\ \frac{1}{2} \end{pmatrix}$ $D = \begin{pmatrix} 100 & 200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Can she prefer A to B and also D to C? Let's see.

outcome	U
\$200	1
\$150	a
\$100	b
\$50	c
\$0	0

Since she prefers D to C, she prefers

$D \succ C$

$EU(D) = \frac{1}{2}\underbrace{U(100)}_{b} + \frac{1}{2}\underbrace{U(200)}_{1} > EU(C) = \underbrace{U(150)}_{a}$

$\frac{1}{2}b + \frac{1}{2}1 > a$ $\boxed{b + 1 > 2a}$

contradicts $2a > 1 + b$

Thus people who are consistently (that is, at every initial level of wealth) risk-averse towards gains and risk-loving towards losses cannot satisfy the axioms of expected utility. If those axioms capture the notion of rationality, then those people are irrational.

VALUE of INFORMATION

The general case (non-monetary outcomes)

probability →	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{2}$	$\frac{1}{4}$
state →	s_1	s_2	s_3	s_4
act ↓				
a	z_1	z_2	z_3	z_4
b	z_5	z_6	z_7	z_8

suppose:

		U utility		V
best	z_8	96	×100	9600
	z_4	80	×100	8000
	z_5	48	×100	4800
	z_1, z_2	32	×100	3200
	z_3, z_6	16	×100	1600
worst	z_7	0	×100	0

then

probability →	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{2}$	$\frac{1}{4}$
state →	s_1	s_2	s_3	s_4
act ↓				
a	32	32	16	80
b	48	16	0	96

$$EU(a) = \frac{1}{16}32 + \frac{3}{16}32 + \frac{1}{2}16 + \frac{1}{4}80 = 2 + 6 + 8 + 20 = \boxed{36} \checkmark$$

3600

$$EU(b) = \frac{1}{16}48 + \frac{3}{16}16 + \frac{1}{4}96 = 3 + 3 + 24 = 30$$

3000

probability →	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{2}$	$\frac{1}{4}$
state →	s_1	s_2	s_3	s_4
act ↓				
a	32	32	16	80
b	48	16	0	96

In the absence of further information.

$\mathbb{E}[U(a)] =$ 36 ←

$\cancel{\mathbb{E}[U(b)] =}$ 30

$\{\{s_1\}, \{s_2\}, \{s_3\}, \{s_4\}\}$

Suppose now that the DM is offered **perfect information for free**.

probability →	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{2}$	$\frac{1}{4}$
state →	s_1	s_2	s_3	s_4
act ↓				
a	32	32	16	80
b	48	16	0	96

- If told s_1 she chooses b and gets utility 48
- If told s_2 she chooses a and gets utility 32
- If told s_3 she chooses a and gets utility 16
- If told s_4 she chooses b and gets utility 96

Her expected utility under free perfect information is $\frac{1}{16}48 + \frac{3}{16}32 + \frac{1}{2}16 + \frac{1}{4}96 =$

$= 3 + 6 + 8 + 24 = \boxed{41}$ 4100

Free perfect information means an **increase in expected utility** of

From 36 to 41 = 5

from 3600 to 4100 = 500

How to monetize the value of information in the general case

probability →	q	$1-q$
state →	s_1	s_2
act ↓		
a	y_1	y_2
b	y_3	y_4

If $y_1 \succsim y_3$, $y_2 \succsim y_4$ and at least one is not $\sim$ then a (weakly) dominates b — rule this out

To avoid triviality let us assume that it is not the case that one act dominates the other.

Assume that

$y_1 \succ y_3$ $\quad$ $y_4 \succ y_2$

$$U(y_1) > U(y_3) \text{ and } U(y_4) > U(y_2)$$

Not enough to tell which act the DM would choose. Assume that he would choose act a:

$$\overbrace{qU(y_1)+(1-q)U(y_2)}^{EU(a)} > \underbrace{qU(y_3)+(1-q)U(y_4)}_{EU(b)}$$

if told s_1 then choose a
if told s_2 choose b

What is the maximum price that the DM would be willing to pay for perfect information?

Each outcome y_i should be thought of a list of all the things that the DM cares about (wealth is just one of them). Separate from each y_i the wealth part and write the outcome as (z_i, W_i) where z_i is that part of y_i that does not refer to the DM's wealth and W_i is the DM's wealth in outcome y_i:

EU with perfect inform.
$qU(y_1)+(1-q)U(y_4)$

$y_1 = (z_1, W_1)$

probability →	q	$1-q$
state →	s_1	s_2
act ↓		
a	(z_1,W_1)	(z_2,W_2)
b	(z_3,W_3)	(z_4,W_4)

$U(z_1,W_1) > U(z_3,W_3)$

If I have to pay \$p for perfect information

Our assumption is that $U(y_1) > U(y_3)$ and $U(y_4) > U(y_2)$ thus

$$U(z_1, W_1) > U(z_3, W_3) \text{ and } U(z_4, W_4) > U(z_2, W_4)$$

What would he choose if, having paid \$$p$ for perfect information, he were informed that the state was s_1? In general, we cannot infer from $U(z_1, W_1) > U(z_3, W_3)$ that $U(z_1, W_1 - p) > U(z_3, W_3 - p)$. Assume this, however and, similarly, $U(z_4, W_4 - p) > U(z_2, W_2 - p)$. Then if informed that S_1 the DM would choose a and if informed that s_2 then he would choose b . Thus with perfect information his expected utility would be

$$q\,U(z_1, W_1 - p) + (1-q)\,U(z_4, W_4 - p) = \overbrace{q\,U(z_1, W_1) + (1-q)\,U(z_2, W_2)}^{\text{without inform.}}$$

The maximum price the DM is willing to pay for perfect information is that value of p that solves the equation:

willing to pay up to solution to ↑ in terms of p

In Chapter 9 of the book (Section 9.3) there is a detailed (more complex) example along these lines.