$$\text{operation } O = \begin{pmatrix} \text{cured} & \text{permanent disability} \\ 90\% & 10\% \end{pmatrix}$$

$$\text{drug treatment } D = \begin{pmatrix} \text{cured} & \text{no benefit} & \text{adverse reaction} \\ 75\% & 10\% & 15\% \end{pmatrix}$$

Which of the two would a risk-averse person choose?

What is the expected value of lottery O?

What is the expected value of lottery D?

meaningless!

Which of the two lotteries is better?

$$\begin{pmatrix} z_1 & z_3 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

$z_2 \quad \begin{pmatrix} z_1 & z_2 & \cdots & z_m \\ 0 & 1 & 0 & 0 \end{pmatrix}$

EXPECTED UTILITY THEORY

$Z = \{z_1, z_2, \ldots, z_m\}$ set of basic outcomes.

$0 \le p_i \le 1 \quad i \in \{1, 2, \ldots, m\}$

A lottery is a probability distribution over Z: $L = \begin{pmatrix} z_1 & z_2 & \ldots & z_m \\ p_1 & p_2 & \ldots & p_m \end{pmatrix}$

$p_1 + p_2 + \cdots + p_m = 1$

Let $\mathcal{L}$ be the set of lotteries. Suppose that the agent has a ranking $\succsim$ of the elements of $\mathcal{L}$:

if $L = \begin{pmatrix} z_1 & z_2 & \ldots & z_m \\ p_1 & p_2 & \ldots & p_m \end{pmatrix}$ and $M = \begin{pmatrix} z_1 & z_2 & \ldots & z_m \\ q_1 & q_2 & \ldots & q_m \end{pmatrix}$ then

$L \succ M$ means that L is considered to be better than M

$L \sim M$ means that L " is just as good as M

Rationality constraints on $\succsim$ (von Neumann-Morgenstern axioms):

… 1. Able to rank basic outcomes (complete and transitive)

2.

$$Z = \{ z_1, z_2, z_3, z_4, z_5 \}$$

best $\quad z_3 \longrightarrow z_{best}$

$\quad z_1, z_4$

$\quad z_2$

worst $\quad z_5 \longrightarrow z_{worst}$

$$L = \begin{pmatrix} z_3 & z_3 \\ p & 1-p \end{pmatrix} \quad \text{compare to} \quad M = \begin{pmatrix} z_3 & z_5 \\ q & 1-q \end{pmatrix}$$

Axiom 2: $\underbrace{L \text{ better than } M}_{L > M}$ if and only if $p > q$

Theorem 1 Let $Z = \{z_1, z_2, \ldots, z_m\}$ be a set of basic outcomes and L the set of lotteries over Z. If $\succsim$ satisfies the von Neumann-Morgenstern axionm then there exists a function $U : Z \to \mathbb{R}$, called a *von Neumann-Morgenstern utility function*, that assigns a number to every basic outcome and is such that, for any two lotteries $L = \begin{pmatrix} z_1 & z_2 & \ldots & z_m \\ p_1 & p_2 & \ldots & p_m \end{pmatrix}$ and $M = \begin{pmatrix} z_1 & z_2 & \ldots & z_m \\ q_1 & q_2 & \ldots & q_m \end{pmatrix}$,

$$L \succ M \qquad \text{if and only if} \qquad \underbrace{p_1 U(z_1) + p_2 U(z_2) + \ldots + p_m U(z_m)}_{\text{expected utility of lottery } L} > \underbrace{q_1 U(z_1) + q_2 U(z_2) + \ldots + q_m U(z_m)}_{\text{expected utility of lottery } M}$$

and

$$L \sim M \qquad \text{if and only if} \qquad \underbrace{p_1 U(z_1) + p_2 U(z_2) + \ldots + p_m U(z_m)}_{\text{expected utility of lottery } L} = \underbrace{q_1 U(z_1) + q_2 U(z_2) + \ldots + q_m U(z_m)}_{\text{expected utility of lottery } M}$$

2.25 3.33

EXAMPLE 1. $Z = \{z_1, z_2, z_3, z_4\}$ $L = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{8} & \frac{5}{8} & 0 & \frac{2}{8} \end{pmatrix}$ $M = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{2}{6} \end{pmatrix}$

Suppose we know that $U = \begin{cases} z_1 & z_2 & z_3 & z_4 \\ 6 & 2 & 8 & 1 \end{cases}$

best z_3
z_1
z_2
worst z_4

Then

$$\mathbb{E}[U(L)] = \frac{1}{8} \cdot 6 + \frac{5}{8} 2 + 0 \cdot 8 + \frac{2}{8} 1 = 2.25$$

$$\mathbb{E}[U(M)] = \frac{1}{6} 6 + \frac{2}{6} 2 + \frac{1}{6} \cdot 8 + \frac{2}{6} 1 = 3.33$$

$$M > L$$

EXAMPLE 2.

$$A = \begin{pmatrix} \text{paid 3-week vacation} & \text{no vacation} \\ 50\% & 50\% \end{pmatrix} \qquad B = \begin{pmatrix} \text{paid 1-week vacation} \\ 100\% \end{pmatrix}$$

Suppose Ann says $\boxed{B \succ A}$ How would she rank

$$C = \begin{pmatrix} \text{paid 3-week vacation} & \text{no vacation} \\ 5\% & 95\% \end{pmatrix} \text{ and } D = \begin{pmatrix} \text{paid 1-week vacation} & \text{no vacation} \\ 10\% & 90\% \end{pmatrix} ?$$

now $C \succ D$

			U
best	3-week vacation	z_1	a
	1-week vacation	z_2	b
worst	no vacation	z_3	c

$a > b > c$

$B \succ A$ then $E[U(B)] > E[U(A)]$

$1 \cdot U(z_2) = U(z_2)$

$= b$

$\frac{1}{2} U(z_1) + \frac{1}{2} U(z_3)$

$= \frac{1}{2} a + \frac{1}{2} c$

$b > \frac{a+c}{2}$ or $\boxed{2b > a+c}$

$$C = \begin{pmatrix} \text{paid 3-week vacation} & \text{no vacation} \\ 5\% & 95\% \end{pmatrix} \text{ and } D = \begin{pmatrix} \text{paid 1-week vacation} & \text{no vacation} \\ 10\% & 90\% \end{pmatrix} ?$$

$C \succ D$ then $E[U(C)] > E[U(D)]$

$$\frac{5}{100} U(z_1) + \frac{95}{100} U(z_3) \qquad \frac{10}{100} U(z_2) + \frac{90}{100} U(z_3)$$

$$= \frac{5}{100} a + \frac{95}{100} c > = \frac{10}{100} b + \frac{90}{100} c$$

$$\frac{5}{100} a + \frac{5}{100} c > \frac{10}{100} b$$

multiply by 100

$$5a + 5c > 10b$$

divide by 5

$$\boxed{a + c > 2b}$$

Contradiction!

Money lotteries

$$L = \begin{pmatrix} \$17 \\ 1 \end{pmatrix} \qquad M = \begin{pmatrix} \$9 & \$25 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$E[M] = \frac{1}{2} 9 + \frac{1}{2} 25 = 17$

$\mathbb{E}[L] =$ 17 $\qquad \mathbb{E}[M] =$ 17

$U(\$25) = \sqrt{25} = 5$

$U(\$17) = \sqrt{17} = 4.12$

$U(\$9) = \sqrt{9} = 3$

Suppose Bob's vNM utility function is: $U(\$x) = \sqrt{x}$

$\mathbb{E}[U(L)] = 1 \cdot \sqrt{17} = 4.12 \quad > \quad \mathbb{E}[U(M)] = \frac{1}{2}\sqrt{9} + \frac{1}{2}\sqrt{25} =$

$\frac{1}{2} 3 + \frac{1}{2} 5 = 4$

RISK AVERSE

$E[A] = 50$ $E[B] = \frac{1}{2}40 + \frac{1}{2}60 = 50$

$$A = \begin{pmatrix} \$0 & \$100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad B = \begin{pmatrix} \$40 & \$60 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$\mathbb{E}[A] =$ 50 $\mathbb{E}[B] =$ 50

Suppose Bob's vNM utility function is: $U(\$x) = \sqrt{x}$

$\mathbb{E}[U(A)] =$ $\frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{100} = \frac{1}{2}0 + \frac{1}{2}10 = 5$

$\mathbb{E}[U(B)] =$ $\frac{1}{2}\sqrt{40} + \frac{1}{2}\sqrt{60} = 7.03$ $B \succ A$

$E[A] = \frac{1}{2}4 + \frac{1}{2}6$

$= 5$

$$A = \begin{pmatrix} \$4 & \$6 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad B = \begin{pmatrix} \$5 \\ 1 \end{pmatrix}$$

	U
\$6	36
\$5	25
\$4	16

$$\mathbf{U(\$x) = x^2}$$

$E[U(A)] = \frac{1}{2}16 + \frac{1}{2}36 = \frac{52}{2}$

$E[U(B)] = 1 \cdot 25 = 25$

$A \succ B$

RISK LOVING

$$L = \begin{pmatrix} \$x_1 & \cdots & \$x_n \\ p_1 & & p_n \end{pmatrix}$$

Re-define attitudes to risk in terms of utility:

Risk-averse if $U(\overbrace{E[L]}^{\text{expected value of } L}) > \underbrace{E[U(L)]}_{\text{expected utility of } L}$

Risk-neutral if $U(E[L]) = E[U(L)]$

Risk-loving if $U(E[L]) < E[U(L)]$