Z = set of basic outcomes = $\{z_1, z_2, \ldots, z_m\}$

Theorem 2. Let $\succsim$ be a von Neumann-Morgenstern ranking of the set of basic lotteries $\mathcal{L}$. Then the following are true.

(A) If $U : Z \to \mathbb{R}$ is a von Neumann-Morgenstern utility function that represents $\succsim$, then, for any two real numbers a and b with $a > 0$, the function $V : Z \to \mathbb{R}$ defined by $V(z_i) = aU(z_i) + b$ $(i = 1,2,\ldots,m)$ is also a von Neumann-Morgenstern utility function that represents $\succsim$.

(B) If $U : Z \to \mathbb{R}$ and $V : Z \to \mathbb{R}$ are two von Neumann-Morgenstern utility functions that represent $\succsim$, then there exist two real numbers a and b with $a > 0$ such that $V(z_i) = aU(z_i) + b$ $(i = 1,2,\ldots,m)$.

$$U = \begin{cases} \begin{array}{cccccc} z_1 & \overset{\text{worst}}{z_2} & \overset{\text{best}}{z_3} & z_4 & \overset{\text{worst}}{z_5} & z_6 \\ 10 & 6 & 16 & 8 & 6 & 14 \end{array} \end{cases}$$

$m = 6$

$\downarrow$ subtract 6 $\quad$ $\left(\begin{array}{c} a = 1 \\ b = -6 \end{array}\right)$

$W: \; 4 \quad 0 \quad 10 \quad 2 \quad 0 \quad 8$

$\downarrow$ multiply by $\frac{1}{10}$ $\quad$ $\left(\begin{array}{c} a = \frac{1}{10} \\ b = 0 \end{array}\right)$

$V: \; \frac{4}{10} \quad \boxed{0} \quad \boxed{1} \quad \frac{2}{10} \quad 0 \quad \frac{8}{10}$ $\quad$ normalized utility function

utility of best outcome is 1

utility of worst outcome is 0

$m = 3$ $\quad$ $Z = \{z_1, z_2, z_3\}$

Question 1 : what is your ranking of Z?

		Suppose utility
best	z_2	1
	z_3	$\frac{4}{7}$
worst	z_1	0

Question 2 : what value of p is such that

$$z_3 = \begin{pmatrix} z_3 \\ 1 \end{pmatrix} \sim \begin{array}{cc} \text{best} & \text{worst} \\ \end{array}\begin{pmatrix} z_2 & z_1 \\ p & 1-p \end{pmatrix}$$

Suppose answer is $p = \frac{4}{7}$

$$EU\uparrow = 1 \cdot U(z_3) = U(z_3) \quad = \quad EU\uparrow = p\, U(z_2) + (1-p)\, U(z_1) = p \cdot 1 + (1-p) \cdot 0 = p$$

$$U(z_3) = \frac{4}{7}$$

$p = \frac{4}{7}$

	Suppose	utility
best	z_2	70
	z_3	46
worst	z_1	14

$$\text{EU of} \begin{pmatrix} z_3 \\ 1 \end{pmatrix} = 1 \cdot U(z_3) = U(z_3)$$

$$\text{EU of} \begin{pmatrix} z_2 & z_1 \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix} = \frac{4}{7} U(z_2) + \frac{3}{7} U(z_1) =$$

$$= \frac{4}{7} \cdot 70 + \frac{3}{7} 14 =$$

$$= 40 + 6 = 46$$

Suppose

best	z_1	1
	z_3	
	z_4	
worst	z_2	0

$$\text{operation } O = \begin{array}{c} \begin{array}{cc} \quad z_1 \quad & \quad z_2 \quad \end{array} \\ \begin{pmatrix} \text{cured} & \text{permanent disability} \\ 90\% & 10\% \end{pmatrix} \end{array}$$

$$\text{drug treatment } D = \begin{array}{c} \begin{array}{ccc} \quad z_1 \quad & \quad z_3 \quad & \quad z_4 \quad \end{array} \\ \begin{pmatrix} \text{cured} & \text{no benefit} & \text{adverse reaction} \\ 75\% & 10\% & 15\% \end{pmatrix} \end{array}$$

What should you do?

N Do nothing: $\begin{pmatrix} z_3 \\ 1 \end{pmatrix} = z_3$

$$\text{operation } O = \begin{array}{cc} z_1 & z_2 \\ \left(\begin{array}{cc} \text{cured} & \begin{array}{c}\text{permanent}\\ \text{disability}\end{array} \\ 90\% & 10\% \end{array}\right) \end{array}$$

$$\text{drug treatment } D = \begin{array}{ccc} z_1 & z_3 & z_4 \\ \left(\begin{array}{ccc} \text{cured} & \begin{array}{c}\text{no}\\ \text{benefit}\end{array} & \begin{array}{c}\text{adverse}\\ \text{reaction}\end{array} \\ 75\% & 10\% & 15\% \end{array}\right) \end{array}$$

	Suppose	normalized utility
best	z_1	1
	z_3	$\frac{95}{100}$
	z_4	$\frac{50}{100}$
worst	z_2	0

Question 2: what value of p is such that $z_3 \sim \begin{array}{cc} \text{best} & \text{worst} \\ \begin{pmatrix} z_1 & z_2 \\ p & 1-p \end{pmatrix} \end{array}$?

Suppose answer is $p = \frac{95}{100}$

Question 3: what value of p is such that $z_4 \sim \begin{pmatrix} z_1 & z_2 \\ p & 1-p \end{pmatrix}$?

Suppose answer is $p = \frac{50}{100}$

$$E[U(N)] = U(z_3) = \frac{95}{100}$$

$$E[U(O)] = \frac{90}{100} U(z_1) + \frac{10}{100} U(z_2) = \frac{90}{100} \cdot 1 + \frac{10}{100} \cdot 0 = \frac{90}{100}$$

$$E[U(D)] = \frac{75}{100} U(z_1) + \frac{10}{100} U(z_3) + \frac{15}{100} U(z_4) =$$

$$= \frac{75}{100} \cdot 1 + \frac{10}{100} \frac{95}{100} + \frac{15}{100} \cdot \frac{50}{100} =$$

$$= ?$$

Compare z_3 to $\begin{pmatrix} z_1 & z_2 \\ p & 1-p \end{pmatrix} = L(p)$

if $p = 1$ $\quad L(1) > z_3$

if $p = 0$ $\quad L(0) < z_3$

Leonard Savage

The Allais paradox

(Maurice Allais, 1952)

$$A = \begin{pmatrix} \$1M \\ 100\% \end{pmatrix} \quad \text{versus} \quad B = \begin{pmatrix} \$2.5M & \$1M & 0 \\ 10\% & 89\% & 1\% \end{pmatrix}$$

$A \succ B$

$$C = \begin{pmatrix} \$1M & 0 \\ 11\% & 89\% \end{pmatrix} \quad \text{versus} \quad D = \begin{pmatrix} \$2.5M & 0 \\ 10\% & 90\% \end{pmatrix}.$$

D > C

	utility
\$ 2.5 M	1
\$ 1 M	a
\$ 0	0

$$A \succ B \quad \text{if and only if} \quad \mathbb{E}[U(A)] > \mathbb{E}[U(B)]$$

$$\underset{a}{\underset{\shortparallel}{\mathbb{E}[U(A)]}} > \underset{\frac{10}{100}1 + \frac{89}{100}a + \frac{1}{100}0}{\underset{\shortparallel}{\mathbb{E}[U(B)]}}$$

$$E[U(C)] < E[U(D)]$$

$$\shortparallel$$

$$\frac{11}{100}a + \frac{89}{100}0 < \frac{10}{100}1 + \frac{90}{100}\cdot 0$$

Contradiction: the value of a cannot satisfy both