MORE THAN TWO CATEGORIES

Enrollment in a class

ECN	*ARE*	*PSY*	*Other*
38%	20%	12%	30%

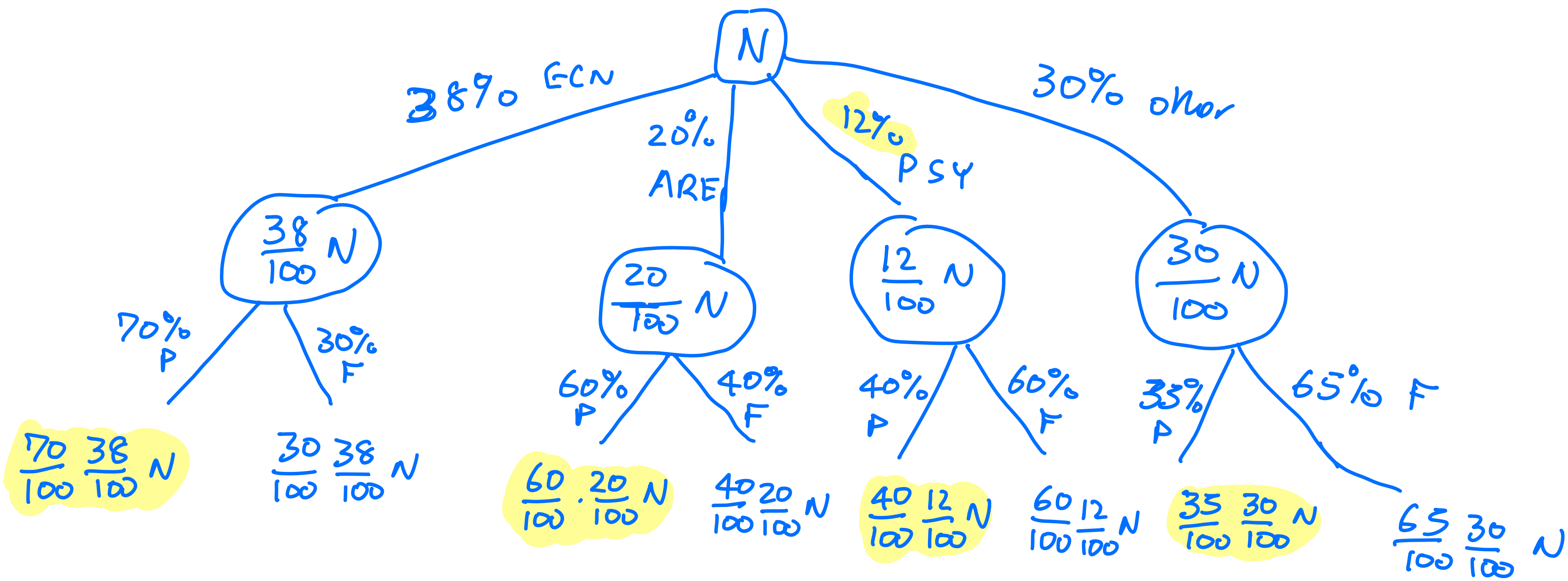
Percentages of those who passed:

major	*ECN*	*ARE*	*PSY*	*Other*
percentage who passed	70%	60%	40%	35%

You learn that Ann passed the class. How likely is it that Ann is a PSY major?

major	*ECN*	*ARE*	*PSY*	*Other*
enrollment	38%	20%	12%	30%
percentage who passed	70%	60%	40%	35%

Ann passed the class. How likely is it that she is a PSY major?



$$P(PSY \mid Passed) = \frac{\frac{40}{100}\frac{12}{100}N}{\frac{70}{100}\frac{38}{100}N + \frac{60}{100}\frac{20}{100}N + \frac{40}{100}\cdot\frac{12}{100}N + \frac{35}{100}\frac{30}{100}N}$$

$$= \frac{48}{539} = 8.9\%$$

Probability and conditional probability

Finite set of ***states*** $S = \{s_1, s_2, ..., s_n\}$. Subsets of *S* are called ***events***.

Probability distribution over *S:*

for all $i = 1, 2, .., n$

$$\begin{array}{cccc} s_1 & s_2 & \dots & s_n \\ p_1 & p_2 & \dots & p_n \end{array}$$

$0 \leq p_i \leq 1$

$p_1 + p_2 + \dots + p_n = 1$

Denote the probability of state *s* by $p(s)$.

Given an event $E \subseteq S$, the probability of *E* is:

$$P(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ \sum_{s \in E} p(s) & \text{if } E \neq \emptyset \end{cases}$$

$\bar{E}$

Denote by $\neg E$ the complement of $E \subseteq S$.

Example

$S = \{a, b, c, d, e, f, g\}$ $\boxed{A = \{a, c, d, e\} \qquad B = \{a, e, g\}}$

$\neg A = \{b, f, g\}$ $\neg B = \{b, c, d, f\}$

Given

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$P(\neg A) = p(b) + p(f) + p(g) = \frac{2}{14} + \frac{1}{14} + \frac{3}{14} = \frac{6}{14} = 1 - P(A)$

$P(A) = p(a) + p(c) + p(d) + p(e) = \frac{1}{14} + 0 + \frac{1}{14} + \frac{6}{14} = \frac{8}{14}$

$P(B) = p(a) + p(e) + p(g) = \frac{1}{14} + \frac{6}{14} + \frac{3}{14} = \frac{10}{14}$

INTERSECTION $A \cap B = \{a, e\}$ $P(A \cap B) = p(a) + p(e) = \frac{1}{14} + \frac{6}{14} = \frac{7}{14}$

UNION $A \cup B = \{a, c, d, e, g\}$ $P(A \cup B) = \frac{1}{14} + 0 + \frac{1}{14} + \frac{6}{14} + \frac{3}{14} = \frac{11}{14}$

Note: for every two events E and F:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

if E and F are disjoint i.e. $E \cap F = \emptyset$ then $P(E \cup F) = P(E) + P(F)$

We denote by $P(E \mid F)$ the probability of E **conditional on** F and define it as:

$$\boxed{P(E \mid F) = \frac{P(E \cap F)}{P(F)}}$$

ASSUMING THAT $P(F) \neq 0$

conditional on F or given F

Continuing the example above where

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$A = \{a, c, d, e\}$ $\quad B = \{a, e, g\}$

$P(A) = \frac{1}{14} + 0 + \frac{1}{14} + \frac{6}{14} = \frac{8}{14}$, $P(B) = \frac{1}{14} + \frac{6}{14} + \frac{3}{14} = \frac{10}{14}$, $P(A \cap B) = \frac{1}{14} + \frac{6}{14} = \frac{7}{14}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{7}{14}}{\frac{10}{14}} = \frac{7}{10}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{7}{14}}{\frac{8}{14}} = \frac{7}{8}$$

The conditional probability formula can also be applied to individual states:

$$\boxed{p(s \mid E) = \begin{cases} 0 & \text{if } s \in E \\ \dfrac{p(s)}{P(E)} & \text{if } s \in E \end{cases}}$$

$\{s\}$

s

$$P(\{s\} \mid C\} = \frac{P(\{s\} \cap C)}{P(C)} = \begin{cases} \dfrac{P(\emptyset)}{P(C)} = \dfrac{0}{P(C)} = 0 & \text{if } s \notin C \\ \dfrac{p(s)}{P(C)} & \text{if } s \in C \end{cases}$$

assuming $P(C) \neq 0$

We can think of $p(\bullet | E)$ as a probability distribution on the entire set S. Continuing the example above

where $S = \{a,b,c,d,e,f,g\}$, $\boxed{A = \{a,c,d,e\}}$ and $\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{array}$ (so that $P(A) = \frac{8}{14}$)

$$p(\bullet | A): \begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{\frac{1}{14}}{\frac{8}{14}} = \frac{1}{8} & 0 & \frac{0}{\frac{8}{14}} = 0 & \frac{1}{8} & \frac{\frac{6}{14}}{\frac{8}{14}} = \frac{6}{8} & 0 & 0 \end{array}$$

For every event E

$$P(E|A) = \sum_{s \in E} p(s|A)$$

Shortcut to obtain the revised or updated probabilities:

Initial or prior probabilities. Note that here they all have the **same denominator**.	$\begin{array}{cccc} a & b & c & d \\ \frac{15}{100} & \frac{70}{100} & \frac{5}{100} & \frac{10}{100} \end{array}$
Information or conditioning event: $\boxed{F = \{a,b,d\}}$	
STEP 1. Set the probability of every state which is not in F to zero:	$\begin{array}{cccc} a & b & c & d \\ & & 0 & \end{array}$
STEP 2. For the other states write new fractions with the same numerators as before:	$\begin{array}{cccc} a & b & c & d \\ \frac{15}{\dots} & \frac{70}{\dots} & 0 & \frac{10}{\dots} \end{array}$
STEP 3. In every denominator put the sum of the numerators: 15+70+10=95. Thus the updated probabilities are:	$\begin{array}{cccc} a & b & c & d \\ \frac{15}{95} & \frac{70}{95} & 0 & \frac{10}{95} \end{array}$

$15 + 70 + 10 = 95$

$p(s|F)$

In the above example, where $\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{array}$ and $\boxed{A = \{a, c, d, e\}}$, to compute $p(\bullet | A)$

Step 1: assign zero probability to states in $\neg A$:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$

Step 2: keep the same numerators for the states in A:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ & 0 & & & & 0 & 0 \end{array}$$

Step 3: since the sum of the numerators is 8, put 8 as the denominator:

$p(s | A)$

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{1}{8} & 0 & \frac{0}{8} & \frac{1}{8} & \frac{6}{8} & 0 & 0 \end{array}$$

$1 + 1 + 6 = 8$

EXAMPLE 2. Sample space or set of states: $\{a, b, c, d, e, f\}$.

Initial or prior probabilities:	$\begin{array}{cccccc} a & b & c & d & e & f \\ \end{array}$ $\begin{pmatrix} \frac{3}{20} & \frac{3}{10} & \frac{1}{20} & 0 & \frac{2}{5} & \frac{1}{10} \end{pmatrix}$
Information:	$\boxed{F = \{a, b, d, e\}}$
STEP 0. Rewrite all the probabilities **with the same denominator:**	$\begin{array}{cccccc} a & b & c & d & e & f \\ \end{array}$ $\begin{pmatrix} \cancel{\frac{3}{20}} & \cancel{\frac{6}{20}} & 0 & & & 0 \end{pmatrix}$
STEP 1. Change the probability of every state which is not in F to zero:	$\begin{array}{cccccc} a & b & c & d & e & f \\ \end{array}$ $\begin{pmatrix} \frac{3}{\ } & \frac{6}{\ } & 0 & \frac{0}{\ } & \frac{8}{\ } & 0 \end{pmatrix}$
STEP 2. Write new fractions which have the same numerators as before:	$\begin{array}{cccccc} a & b & c & d & e & f \\ \end{array}$ $\begin{pmatrix} & & 0 & & & 0 \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 3+6+8=17. $p(s \mid F)$	$\begin{array}{cccccc} a & b & c & d & e & f \\ \end{array}$ $\begin{pmatrix} \frac{3}{17} & \frac{6}{17} & 0 & 0 & \frac{8}{17} & 0 \end{pmatrix}$

$$\begin{array}{cccccc} a & b & c & d & e & F \\ \frac{3}{20} & \frac{6}{20} & \frac{1}{20} & 0 & \frac{8}{20} & \frac{2}{20} \end{array}$$

$3+6+8 = 17$

for any event A

$$P(A \mid F) = \left\langle \begin{array}{c} \dfrac{P(A \cap F)}{P(F)} \\ \\ \displaystyle\sum_{s \in A} p(s \mid F) \end{array} \right\rangle$$

these two are equal