MORE THAN TWO CATEGORIES

Enrollment in a class

ECN	*ARE*	*PSY*	*Other*
38%	20%	12%	30%

Percentages of those who passed:

major	*ECN*	*ARE*	*PSY*	*Other*
percentage who passed	70%	60%	40%	35%

You learn that Ann passed the class. How likely is it that Ann is a PSY major?

major	*ECN*	*ARE*	*PSY*	*Other*	**Ann passed the class. How likely is it that she is a PSY major?**
enrollment	38%	20%	12%	30%	
percentage who passed	70%	60%	40%	35%	

Probability and conditional probability

Finite set of ***states*** $S = \{s_1, s_2, ..., s_n\}$. Subsets of *S* are called ***events***.

Probability distribution over *S:*

$$\begin{matrix} s_1 & s_2 & \cdots & s_n \\ p_1 & p_2 & \cdots & p_n \end{matrix}$$

Denote the probability of state *s* by $p(s)$.

Given an event $E \subseteq S$, the probability of *E* is:

$$P(E) = \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

Denote by $\neg E$ the complement of $E \subseteq S$.

Example

$S = \{a, b, c, d, e, f, g\}$ $A = \{a, c, d, e\}$ $B = \{a, e, g\}$

$\neg A =$ $\neg B =$

Given

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$P(A) =$ $P(B) =$

$A \cap B =$ $P(A \cap B) =$

$A \cup B =$ $P(A \cup B) =$.

Note: for every two events E and F:

$$P(E \cup F) =$$

We denote by $P(E \mid F)$ the probability of E **conditional on** F and define it as:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Continuing the example above where

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$A = \{a, c, d, e\}$ $\qquad$ $B = \{a, e, g\}$

$P(A) = \frac{1}{14} + 0 + \frac{1}{14} + \frac{6}{14} = \frac{8}{14}$, $\quad P(B) = \frac{1}{14} + \frac{6}{14} + \frac{3}{14} = \frac{10}{14}$, $\quad P(A \cap B) = \frac{1}{14} + \frac{6}{14} = \frac{7}{14}$

$$P(A \mid B) =$$

$$P(B \mid A) = \qquad .$$

The conditional probability formula can also be applied to individual states:

$$p(s \mid E) = \begin{cases} \quad & \text{if} \\ \quad & \text{if} \end{cases}$$

We can think of $p(\bullet|E)$ as a probability distribution on the entire set S. Continuing the example above

where $S=\{a,b,c,d,e,f,g\}$, $A=\{a,c,d,e\}$ and $\begin{matrix} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{matrix}$ (so that $P(A)=\frac{8}{14}$)

$$\begin{matrix} & a & b & c & d & e & f & g \\ p(\bullet|A): & & & & & & & \end{matrix}$$

Shortcut to obtain the revised or updated probabilities:

Initial or prior probabilities. Note that here they all have the **same denominator**.	$\begin{pmatrix} a & b & c & d \\ \frac{15}{100} & \frac{70}{100} & \frac{5}{100} & \frac{10}{100} \end{pmatrix}$
Information or conditioning event: $F=\{a,b,d\}$	
STEP 1. Set the probability of every state which is not in *F* to zero:	$\begin{pmatrix} a & b & c & d \\ & & 0 & \end{pmatrix}$
STEP 2. For the other states write new fractions with the same numerators as before:	$\begin{pmatrix} a & b & c & d \\ \frac{15}{\dots} & \frac{70}{\dots} & 0 & \frac{10}{\dots} \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 15+70+10=95. Thus the updated probabilities are:	$\begin{pmatrix} a & b & c & d \\ \frac{15}{95} & \frac{70}{95} & 0 & \frac{10}{95} \end{pmatrix}$

In the above example, where $\begin{matrix} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{matrix}$ and $A = \{a, c, d, e\}$, to compute $p(\bullet | A)$

Step 1: assign zero probability to states in $\neg A$:

$$\begin{matrix} a & b & c & d & e & f & g \end{matrix}$$

Step 2: keep the same numerators for the states in *A*:

$$\begin{matrix} a & b & c & d & e & f & g \\ & 0 & & & & 0 & 0 \end{matrix}$$

Step 3: since the sum of the numerators is 8, put 8 as the denominator:

$$\begin{matrix} a & b & c & d & e & f & g \\ \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{6}{8} & 0 & 0 \end{matrix}$$

EXAMPLE 2. Sample space or set of states: $\{a, b, c, d, e, f\}$.

Initial or prior probabilities:	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{20} & \frac{3}{10} & \frac{1}{20} & 0 & \frac{2}{5} & \frac{1}{10} \end{pmatrix}$
Information:	$F = \{a, b, d, e\}$
STEP 0. Rewrite all the probabilities **with the same denominator:**	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \end{pmatrix}$
STEP 1. Change the probability of every state which is not in F to zero:	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \end{pmatrix}$
STEP 2. Write new fractions which have the same numerators as before:	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 3+6+8=17.	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{17} & \frac{6}{17} & 0 & 0 & \frac{8}{17} & 0 \end{pmatrix}$

INDEPENDENT EVENTS.

We say that two events *A* and *B* are independent if

$$P(A \cap B) = P(A)\,P(B) \qquad (*)$$

It follows from this and the definition of conditional probability that if *A* and *B* are independent then

$$P(A \mid B) = P(A) \text{ and } P(B \mid A) = P(B) \qquad (**)$$

Alternatively, one can take one of the two equalities in (**) as definition of independence and derive both the other and (*). Thus (*) and (**) are equivalent.

Going back to our example where $S=\{a,b,c,d,e,f,g\}$, $A=\{a,c,d,e\}$, $B=\{a,e,g\}$, $A\cap B=\{a,e\}$ and

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$P(A)=$, $P(B)=$, $P(A\cap B)=$ $P(A)P(B)=$

On the other hand, if $S=\{a,b,c,d,e,f,g,h,i\}$ and

a	b	c	d	e	f	g	h	i
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$

Then $E=\{a,b,c,e\}$ and $F=\{c,d,e,g\}$ are independent. In fact, $P(E)=$, $P(F)=$,
$E\cap F=\{c,e\}$, $P(E\cap F)=$ and thus $P(E\cap F)=P(E)P(F)$.

Bayes' formula

Let E and F be two events such that $P(E) > 0$ and $P(F) > 0$. Then

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \tag{1}$$

and

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)} \tag{2}$$

From (2) we get that

(3)

Substituting (3) into (1) we get

Bayes' formula (4)

Bayes' theorem

Bayes' formula says that $\boxed{P(E \mid F) = \frac{P(F \mid E)\,P(E)}{P(F)}}$. From set theory we have that, given any two sets *A* and *B*,

$$A = \left(A \cap B\right) \cup \left(A \cap \neg B\right) \qquad (5)$$

and the two sets $\left(A \cap B\right)$ and $\left(A \cap \neg B\right)$ are disjoint. Thus $P(A) = P\left(A \cap B\right) + P\left(A \cap \neg B\right)$.

Hence in the denominator of Bayes' formula we can replace $P(F)$ with

Then, using conditional probability we get that $P\left(F \cap E\right) =$ and

$P\left(F \cap \neg E\right) =$

Thus $P(F) =$.

Replacing this in Bayes' formula we get

Bayes' theorem (6)

$$P(E \mid F) = \frac{P(F \mid E)\,P(E)}{P(F \mid E)P(E) + P(F \mid \neg E)P(\neg E)}$$

EXAMPLE.

Enrollment in a class is as follows: 60% econ majors (*E*), 40% other majors ($\neg E$). In the past, 80% of the econ majors passed and 65% of the other majors passed. A student tells you that she passed the class. What is the probability that she is an econ major? Let *P* stand for “Pass the class”.

$$P(E \mid P) =$$