INDEPENDENT EVENTS.

We say that two events *A* and *B* are independent if

$$P(A \cap B) = P(A)\,P(B) \qquad (*)$$

It follows from this and the definition of conditional probability that if *A* and *B* are independent then

$$P(A \mid B) = P(A) \text{ and } P(B \mid A) = P(B) \qquad (**)$$

Alternatively, one can take one of the two equalities in (**) as definition of independence and derive both the other and (*). Thus (*) and (**) are equivalent.

Going back to our example where $S=\{a,b,c,d,e,f,g\}$, $A=\{a,c,d,e\}$, $B=\{a,e,g\}$, $A\cap B=\{a,e\}$ and

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$P(A)=$, $P(B)=$, $P(A\cap B)=$ $P(A)P(B)=$

On the other hand, if $S=\{a,b,c,d,e,f,g,h,i\}$ and

a	b	c	d	e	f	g	h	i
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$

Then $E=\{a,b,c,e\}$ and $F=\{c,d,e,g\}$ are independent. In fact, $P(E)=$, $P(F)=$,

$E\cap F=\{c,e\}$, $P(E\cap F)=$ and thus $P(E\cap F)=P(E)P(F)$.

Bayes' formula

Let E and F be two events such that $P(E) > 0$ and $P(F) > 0$. Then

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \qquad (1)$$

and

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)} \qquad (2)$$

From (2) we get that

(3)

Substituting (3) into (1) we get

Bayes' formula (4)

Bayes' theorem

Bayes' formula says that $P(E\,|\,F)=\dfrac{P(F\,|\,E)\,P(E)}{P(F)}$. From set theory we have that, given any two sets *A* and *B*,

$$A=\left(A\cap B\right)\cup\left(A\cap\neg B\right) \qquad (5)$$

and the two sets $\left(A\cap B\right)$ and $\left(A\cap\neg B\right)$ are disjoint. Thus $P(A)=P\left(A\cap B\right)+P\left(A\cap\neg B\right)$.

Hence in the denominator of Bayes' formula we can replace $P(F)$ with

Then, using conditional probability we get that $P\left(F\cap E\right)=$ and

$P\left(F\cap\neg E\right)=$

Thus $P(F)=$.

Replacing this in Bayes' formula we get

Bayes' theorem (6)

$$P(E \mid F) = \frac{P(F \mid E)\, P(E)}{P(F \mid E)P(E) + P(F \mid \neg E)P(\neg E)}$$

EXAMPLE.

Enrollment in a class is as follows: 60% econ majors (E), 40% other majors ($\neg E$). In the past, 80% of the econ majors passed and 65% of the other majors passed. A student tells you that she passed the class. What is the probability that she is an econ major? Let P stand for "Pass the class".

$$P(E \mid P) =$$

Back to previous examples

EXAMPLE 1. Testing for a disease

Base rate of a disease: percentage of the population that has the disease

Sensitivity of a test: percentage of those who have the disease that tests positive

Specificity of a test: percentage of those who **do not** have the disease that tests negative

Suppose:

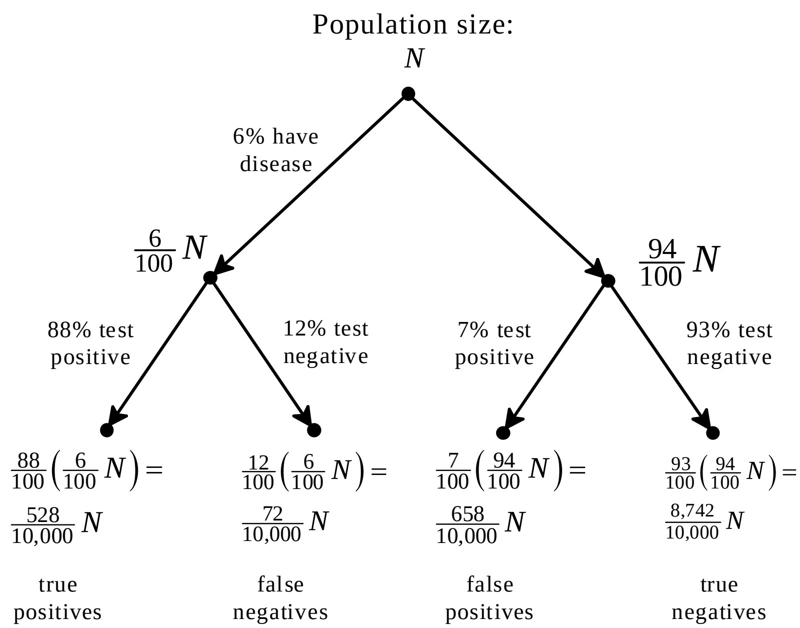
Base rate = 6%

Sensitivity = 88%

Specificity = 93%

Suppose you test positive. What is the probability that you have the disease?

Previous analysis:



The probability of having the disease, conditional on testing positive is:

$$\frac{528\frac{N}{10,000}}{528\frac{N}{10,000}+658\frac{N}{10,000}}=\frac{528\frac{N}{10,000}}{1,186\frac{N}{10,000}}=\frac{528}{1,186}=0.4452=44.52\%$$

D = have the disease $\neg D =$ do not have disease

$+$ = test positive $-$ = test negative

Base rate = 6%

Sensitivity = 88%

Specificity = 93%

By Bayes' rule:

$$P(D \mid +) =$$

EXAMPLE 2. More than two categories

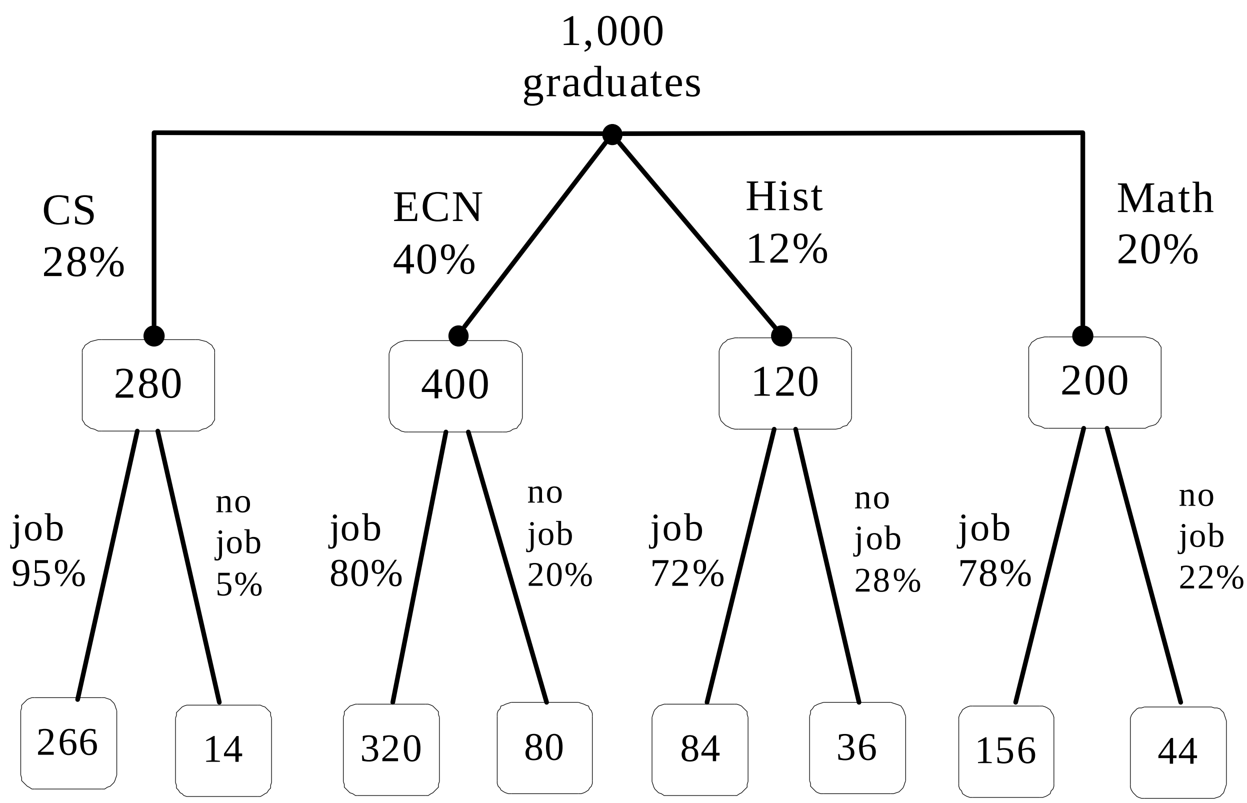
Base rates of seniors who graduated within the past 6 months:

CS	*Econ*	*Hist*	*Math*
28%	40%	12%	20%

Percentages of those who found a job within 6 months of graduation by major:

Major:	*CS*	*Econ*	*Hist*	*Math*
% who found a job:	95%	80%	70%	78%

You learn that Ann graduated 6 months ago and has already found a job. What is the probability that Ann is an Econ major?



The **probability of Ann being and Econ major, given that she found a job is thus**:

$$\frac{320}{266+320+84+156}=\frac{320}{826}=0.3874=38.74\%$$

CS	$Econ$	$Hist$	$Math$
28%	40%	12%	20%

Major:	CS	$Econ$	$Hist$	$Math$
% who found a job:	95%	80%	70%	78%

Now we need a version of Bayes' rule that allows for more than two conditioning events.

Let S be the set of states and $\{E_1, E_2, ..., E_m\}$ be a partition of S, that is,

- $E_1 \cup E_2 \cup ... \cup E_m = S$
- For all $i, j \in \{1, 2, ...m\}$ with $i \neq j$, $E_i \cap E_j = \varnothing$

Let $F \subseteq S$ be an arbitrary event. Then

$F = (F \cap E_1) \cup (F \cap E_2) \cup ... \cup (F \cap E_m)$, all disjoint events. Thus

$$P(F) = \underbrace{P\left(F \cap E_1\right)} + \underbrace{P\left(F \cap E_2\right)} + ... + \underbrace{P\left(F \cap E_m\right)}$$

Hence, $P(E_i \mid F) =$

CS	*Econ*	*Hist*	*Math*
28%	40%	12%	20%

and, by hypothesis, CS, ECN, HIS, MAT is an exhaustive list of majors:

$$P(CS) = \frac{28}{100}, \quad P(ECN) = \frac{40}{100}, \quad P(HIS) = \frac{12}{100}, \quad P(MAT) = \frac{20}{100}$$

Major:	*CS*	*Econ*	*Hist*	*Math*
% who found a job:	95%	80%	70%	78%

$$P(J \mid CS) = \frac{95}{100}, \quad P(J \mid ECN) = \frac{80}{100}, \quad P(J \mid HIS) = \frac{70}{100}, \quad P(J \mid MAT) = \frac{78}{100}$$

$$P(\neg J \mid CS) = \frac{5}{100}, \quad P(\neg J \mid ECN) = \frac{20}{100}, \quad P(\neg J \mid HIS) = \frac{30}{100}, \quad P(\neg J \mid MAT) = \frac{12}{100}$$

Thus

$$P(ECN \mid J) =$$

One more example:

An exam consisted of 3 questions: Questions 1 and 2 were very difficult while Question 3 was very easy. Three students, A, B and C took the exam. The TA informs the professor that everybody answered Question 3, but only one students answered Question 1 and only one student (possibly the same) answered Question 2. Student A is the best student of the three, but not by far.

1. What is the set of states?

2. What probabilities does the professor assign to the states? Suppose:

(A,A)	(A,B)	(A,C)	(B,A)	(B,B)	(B,C)	(C,A)	(C,B)	(C,C)
$\frac{8}{54}$	$\frac{6}{54}$	$\frac{6}{54}$	$\frac{8}{54}$	$\frac{6}{54}$	$\frac{6}{54}$	$\frac{6}{54}$	$\frac{6}{54}$	$\frac{2}{54}$

3. The TA now informs the professor that that student A did **not** answer Question 1.
   What is the probability that student A answered Question 2?

Let A1 be the event that student A answered Question 1 and A2 the event that student A answered Question 2. What we are looking for is

$$P(A2 \mid \neg A1)$$

What formula should we use?

(A,A)	(A,B)	(A,C)	(B,A)	(B,B)	(B,C)	(C,A)	(C,B)	(C,C)
$\frac{8}{54}$	$\frac{6}{54}$	$\frac{6}{54}$	$\frac{8}{54}$	$\frac{6}{54}$	$\frac{6}{54}$	$\frac{6}{54}$	$\frac{6}{54}$	$\frac{2}{54}$