Exponential or discounted utility

Time consistency of preferences

date	0	1	2	3
Plan A	–	x	y	z
Plan B	–	y	z	x

Suppose that you "choose" Plan *B*:

$$\text{Plan } B \succ_0 \text{Plan } A$$

at time 0

Now when date 1 comes along you re-examine those two plans and are free to change your mind (there was no commitment). Your preferences are **time consistent** if at date 1 you maintain the same ranking that you had at time 0:

$$\text{Plan } B \succ_1 \text{Plan } A$$

Recall

$t \geq 0$

$$U_0(z,t) = \delta^t u_t(z) \qquad t = t - 0$$

$s=0$

Extend this to the preferences at any time s:

$$\boxed{U_s(z,t) = \delta^{t-s} u_t(z) \quad \text{assuming that} \quad t \geq s}$$

$$U_0(z,4) = \delta^4 u_4(z)$$
$$U_1(z,4) = \delta^3 u_4(z)$$
$$U_2(z,4) = \delta^2 u_4(z)$$
$$U_3(z,4) = \delta^1 u_4(z)$$
$$U_4(z,4) = \delta^0 u_4(z) = u_4(z)$$

$$U_s(z,t) = \delta^{t-s} u_t(z) \qquad \text{assuming that } t \geq s$$

	Date 0	Date 1	Date 2	Date 3	Date 4
Plan A	--	--	x	y	x
Plan B	--	--	y	z	x

$$U_0(\text{Plan A}) = \delta^2 u_2(x) + \delta^3 u_3(y) + \delta^4 u_4(x)$$

$$U_1(\text{Plan A}) = \delta u_2(x) + \delta^2 u_3(y) + \delta^3 u_4(x)$$

$$U_2(\text{Plan A}) = u_2(x) + \delta u_3(y) + \delta^2 u_4(x)$$

And similarly for the utility of Plan B.

Now suppose that at time 0 you prefer Plan A to Plan B:

$$\underbrace{\delta^2 u_2(x) + \delta^3 u_3(y) + \delta^4 u_4(x)}_{U_0(\text{Plan A})} > \underbrace{\delta^2 u_2(y) + \delta^3 u_3(z) + \delta^4 u_4(x)}_{U_0(\text{Plan B})} \quad (**)$$

Divide both sides of (**) by δ :

$$\underbrace{\delta u_2(x) + \delta^2 u_3(x) + \delta^3 u_4(x)}_{U_1(\text{Plan A})} > \underbrace{\delta u_2(y) + \delta^2 u_3(z) + \delta^3 u_4(x)}_{U_1(\text{Plan B})}$$

Divide both sides of (**) by δ^2 :

$$\underbrace{u_2(x) + \delta u_3(y) + \delta^2 u_4(x)}_{U_2(\text{Plan A})} > \underbrace{u_2(y) + \delta u_3(z) + \delta^2 u_4(x)}_{U_2(\text{Plan B})}$$

The hyperbolic utility model (the β-δ model)

Suppose that on January 1, 2024 you were offered either

- \$1,000 to be collected on January 1, 2025 (12 months later), or
- \$1,500 to be collected on May 1, 2025 (16 months later).

What would you choose?

\$1,500 on May 1 > \$1,000 on Jan. 1

Suppose that you are asked again on January 1, 2025: what do you choose:

- \$1,000 to be collected now or
- \$1,500 to be collected 4 months from now (on May 1, 2025)

\$1,000 now > \$1,500 4 months from now

Recall that in the **discounted (or exponential) utility** model

$$U_0(z,t) = \underbrace{\delta^t u_t(z)}_{} = \begin{cases} u_0(z) & \text{if } t = 0 \\ \delta^t u_t(z) & \text{if } t > 0 \end{cases} \qquad (*)$$

where $0 < \delta \leq 1$ is the *discount factor*.

In the **hyperbolic utility model**

$$U_0(z,t) = \cancel{\delta^t u_t(z)} = \begin{cases} u_0(z) & \text{if } t = 0 \\ \beta\,\delta^t u_t(z) & \text{if } t > 0 \end{cases} \qquad (**)$$

with $0 < \beta \leq 1$

If $\beta = 1$ then Exponential utility = hyperbolic utility

If $\beta < 1$ then they are different

discounted utility model: $U_s(z,t) = \begin{cases} u_s(z) & \text{if } t = s \\ \delta^{t-s} u_t(z) & \text{if } t > s \end{cases}$

hyperbolic utility model: $U_s(z,t) = \begin{cases} u_s(z) & \text{if } t = s \\ \beta \delta^{t-s} u_t(z) & \text{if } t > s \end{cases}$

EXAMPLE 1.

↓

	Date 0	Date 1	Date 2	Date 3
Plan A	--	--	$\cancel{x}$ 6	$\cancel{y}$ 0
Plan B	--	--	$\cancel{w}$ 1.5	$\cancel{z}$ 9

Suppose $u_2(x) = 6,\ u_3(y) = 0,\ u_2(w) = 1.5,\ u_3(z) = 9$ $\beta = 0.6$ and $\delta = 0.8$
Then

$U_0(\text{Plan A}) = \beta\delta^2 6 + \beta\delta^3 \cdot 0 = (0.6)(0.8)^2 \cdot 6 = 2.3$

$U_0(\text{Plan B}) = \beta\delta^2(1.5) + \beta\delta^3 \cdot 9 + = (0.6)(0.8)^2(1.5) + (0.6)(0.8)^3 \cdot 9 = 3.34$

Plan B $\succ_0$ Plan A

Now consider preferences at date 2:

$U_2(\text{Plan A}) = u_2(x) + \beta\delta^1 u_3(y) = 6 + (0.6)(0.8)\cdot 0 = 6$

$U_2(\text{Plan B}) = u_2(w) + \beta\delta^1 u_3(z) = 1.5 + (0.6)(0.8)\cdot 9 = 5.82$

Plan A $\succ_2$ Plan B

time inconsistent