How to aggregate the preferences of a group of individuals

X set of alternatives that society has to choose from.

N $N = \{1, 2, \ldots, n\}$ set of individuals

For every $i \in N$, $\quad \succsim_i \quad$ i's preference relation over X complete and transitive

- complete: for every $x, y \in X$, either $x \succsim_i y$ or $y \succsim_i x$ or both
- transitive: For every $x, y, z \in X$ if $x \succsim_i y$ and $y \succsim_i z$ then $x \succsim_i z$

$x \succsim_i y$ $\quad$ i considers x to be at least as good as y

$x \succ_i y$ $\quad$ i considers x to be better than y

$x \sim_i y$ $\quad$ i " " just as good as y

Issue: how to aggregate the preferences of the individuals into a single ranking that can be viewed as "society's ranking".

$\succsim$ $\quad$ (without subscript) society's preference relation over X

$x \succsim y$ $\quad$ for the group x is at least as good as y

$x \succ y$ $\quad$ better than

$x \sim y$ $\quad$ just as good as

function f: $\quad (\succsim_1, \succsim_2, \ldots, \succsim_n) \mapsto \succsim$

social preference function

5 people, 2 alternatives

1: $A \succ_1 B$

2: $A \succ_2 B$

3: $B \succ_3 A$

4: $A \sim_4 B$

5: $A \sim_5 B$

$\#(A \succ B) = 2$

$\#(B \succ A) = 1$

Majority rule says:

$A \succ B$

$\#(x \succ y)$: number of people for whom x is better than y

$\#(y \succ x)$: " " y is better than x

Majority rule:
- if $\#(x \succ y) > \#(y \succ x)$ then declare $x \succ y$
- otherwise declare $x \sim y$

Majority rule

Let $X = \{A, B, C\}$ and $S = \{1, 2, 3\}$ and

	1's ranking	2's ranking	3's ranking
best	A	C	B
	B	A	C
worst	C	B	A

$\#(A \succ B) = 2$, $\#(B \succ A) = 1$ so declare $A \succ B$

$\#(B \succ C) = 2$, $\#(C \succ B) = 1$ " $B \succ C$

$\#(C \succ A) = 2$, $\#(A \succ C) = 1$ " $C \succ A$

Problem 1: $\succ$ not transitive

Problem 2: can be manipulated. Suppose Individual 2 sets the agenda ...

Suppose individual 2 sets the agenda

First vote between A and B $\rightsquigarrow$ A by majority

Second vote between A (the winner of vote 1) and C

$\rightsquigarrow$ C by majority

In his 1951 Ph.D. thesis Kenneth Arrow asked: what is a good *social preference function* (or aggregation rule)?

$$\text{function } f: \quad (\succsim_1, \succsim_2, \ldots, \succsim_n) \mapsto \succsim$$

There are MANY possible social preference functions

E.g. let $X = \{A, B\}$ and $S = \{1, 2\}$

possible rankings of Individual 1: $A \succ_1 B$, $B \succ_1 A$, $A \sim_1 B$

possible rankings of Individual 2: $A \succ_2 B$, $B \succ_2 A$, $A \sim_2 B$

Thus 9 possible profiles of preferences:

		Individual 2's ranking		
		$A \succ_2 B$	$A \sim_2 B$	$B \succ_2 A$
Individual	$A \succ_1 B$			
1's	$A \sim_1 B$			
ranking	$B \succ_1 A$			

need to have either $A \succ B$ or $A \sim B$ or $B \succ A$

How many different ways to fill these 9 boxes?

i.e. How many different social preference functions?

$3^9 = 19{,}683$

One of them is:
if 1 and 2 agree that x is better than y then $x \succ y$, otherwise $x \sim y$

		Individual 2's ranking		
		$A \succ_2 B$	$A \sim_2 B$	$B \succ_2 A$
Individual	$A \succ_1 B$	$A \succ B$	$A \sim B$	$A \sim B$
1's	$A \sim_1 B$	$A \sim B$	$A \sim B$	$A \sim B$
ranking	$B \succ_1 A$	$A \sim B$	$A \sim B$	$B \succ A$

Second example: $X = \{A, B, C\}$ and $S = \{1, 2, 3\}$

and only strict rankings can be reported:

$$A \succ B \succ C \qquad 2$$

$$A \succ C \succ B$$

$$B \succ A \succ C \qquad 1$$

$$B \succ C \succ A$$

$$C \succ A \succ B \qquad 3$$

$$C \succ B \succ A$$

An input is a triple of strict rankings $(\succ_1, \succ_2, \succ_3)$

How many possible inputs? $(\underset{1}{\square}, \underset{2}{\square}, \underset{3}{\square})$

$6^3 = 216$

SPF $\square$ $\square$ $\square$ . . - . $\square$ Fill in 216 boxes

one of 6 rankings

6^{216} possible SPFs $\qquad 6^{216} = 1.2 \cdot 10^{168}$

What is a good or reasonable SPF?

Establish some principles or *desiderata* or axioms

Example: $X = \{A, B\}$, $S = \{1, 2\}$
and only strict rankings: $A \succ B$ or $B \succ A$

Then 4 possible profiles and 16 possible functions:

profile $\rightarrow$ SPF $\downarrow$	$A{\succ_1}B$ $A{\succ_2}B$	$A{\succ_1}B$ $B{\succ_2}A$	$B{\succ_1}A$ $A{\succ_2}B$	$B{\succ_1}A$ $B{\succ_2}A$
~~SPF - 1~~	$\cancel{A \succ B}$	$\cancel{A \succ B}$	$\cancel{A \succ B}$	$\cancel{A \succ B}$
SPF - 2	$A \succ B$	$A \succ B$	$A \succ B$	$B \succ A$
~~SPF - 3~~	$\cancel{A \succ B}$	$\cancel{A \succ B}$	$\cancel{B \succ A}$	$\cancel{A \succ B}$
SPF - 4	$A \succ B$	$A \succ B$	$B \succ A$	$B \succ A$
~~SPF - 5~~	$\cancel{A \succ B}$	$\cancel{B \succ A}$	$\cancel{A \succ B}$	$\cancel{A \succ B}$
SPF - 6	$A \succ B$	$B \succ A$	$A \succ B$	$B \succ A$
~~SPF - 7~~	$\cancel{A \succ B}$	$\cancel{B \succ A}$	$\cancel{B \succ A}$	$\cancel{A \succ B}$
SPF - 8	$A \succ B$	$B \succ A$	$B \succ A$	$B \succ A$
SPF - 9	$B \succ A$	$A \succ B$	$A \succ B$	$A \succ B$
SPF - 10	$B \succ A$	$A \succ B$	$A \succ B$	$B \succ A$
SPF - 11	$B \succ A$	$A \succ B$	$B \succ A$	$A \succ B$
SPF - 12	$B \succ A$	$A \succ B$	$B \succ A$	$B \succ A$
SPF - 13	$B \succ A$	$B \succ A$	$A \succ B$	$A \succ B$
SPF - 14	$B \succ A$	$B \succ A$	$A \succ B$	$B \succ A$
SPF - 15	$B \succ A$	$B \succ A$	$B \succ A$	$A \succ B$
SPF - 16	$B \succ A$	$B \succ A$	$B \succ A$	$B \succ A$

Unanimity requirement

Good property: if both say $x > y$ then for society $x > y$

UNANIMITY

By appealing to Unanimity we can discard all except:

profile $\rightarrow$ SPF $\downarrow$	$A \succ_1 B$ $A \succ_2 B$	$A \succ_1 B$ $B \succ_2 A$	$B \succ_1 A$ $A \succ_2 B$	$B \succ_1 A$ $B \succ_2 A$
SPF - 2	$A \succ B$	$A \succ B$	$A \succ B$	$B \succ A$
~~SPF - 4~~	~~$A \succ B$~~	~~$A \succ B$~~	~~$B \succ A$~~	~~$B \succ A$~~
~~SPF - 6~~	~~$A \succ B$~~	~~$B \succ A$~~	~~$A \succ B$~~	~~$B \succ A$~~
SPF - 8	$A \succ B$	$B \succ A$	$B \succ A$	$B \succ A$

individual 1 is a dictator

individual 2 is a dictator

NON-DICTATORHIP

$\downarrow$

profile $\rightarrow$ SPF $\downarrow$	$A \succ_1 B$ $A \succ_2 B$	$A \succ_1 B$ $B \succ_2 A$	$B \succ_1 A$ $A \succ_2 B$	$B \succ_1 A$ $B \succ_2 A$
SPF - 2	$A \succ B$	$A \succ B$	$A \succ B$	$B \succ A$
SPF - 8	$A \succ B$	$B \succ A$	$B \succ A$	$B \succ A$

Arrow's axioms

- **Axiom 1: Unrestricted Domain or Freedom of Expression**

  At the individual level, any complete and transitive ranking should be allowed.

- **Axiom 2: Rationality**

  Also the social ranking should be complete and transitive