Set of alternatives among which society has to choose:

$$X = \{x_1, x_2, \ldots, x_m\}$$

Set of individuals (members of society or voters:

$$S = \{1, 2, \ldots, n\}$$

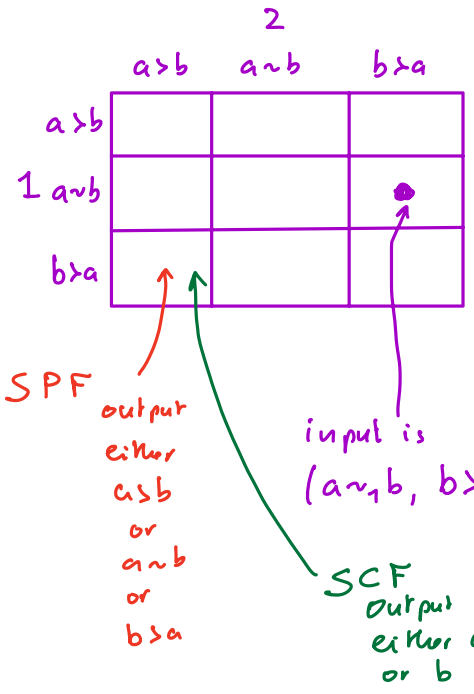
Each voter i has a complete and transitive ranking $\succsim_i$ of X

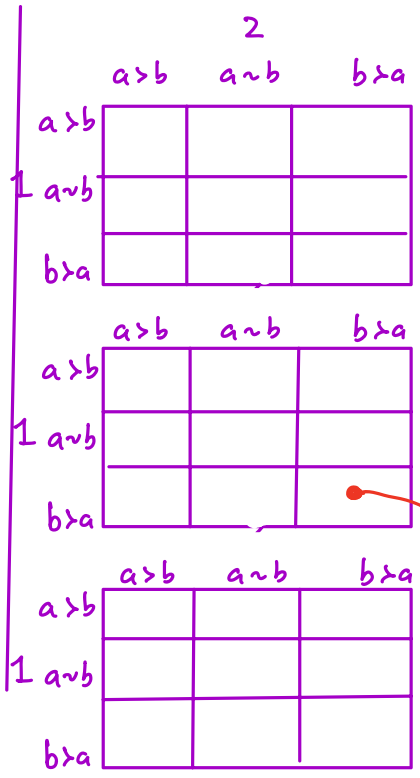
Social preference function: $\underbrace{(\succsim_1, \succsim_2, \ldots, \succsim_n)}_{input} \mapsto \underbrace{\succsim}_{output}$ ranking of X

Social choice function: $\underbrace{(\succsim_1, \succsim_2, \ldots, \succsim_n)}_{input} \mapsto \underbrace{x \in X}_{output}$ one of the alternatives in X

2 alternatives, 3 voters

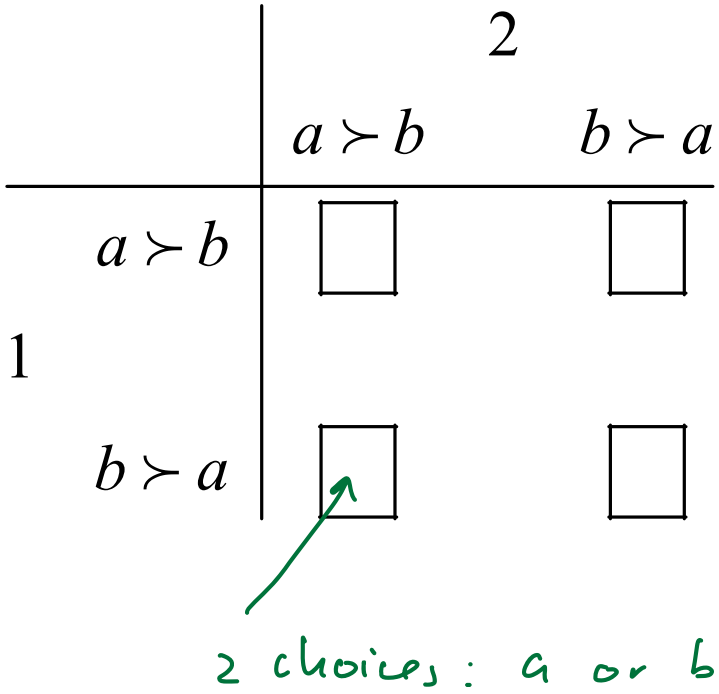
2 alternatives, a and b
2 voters





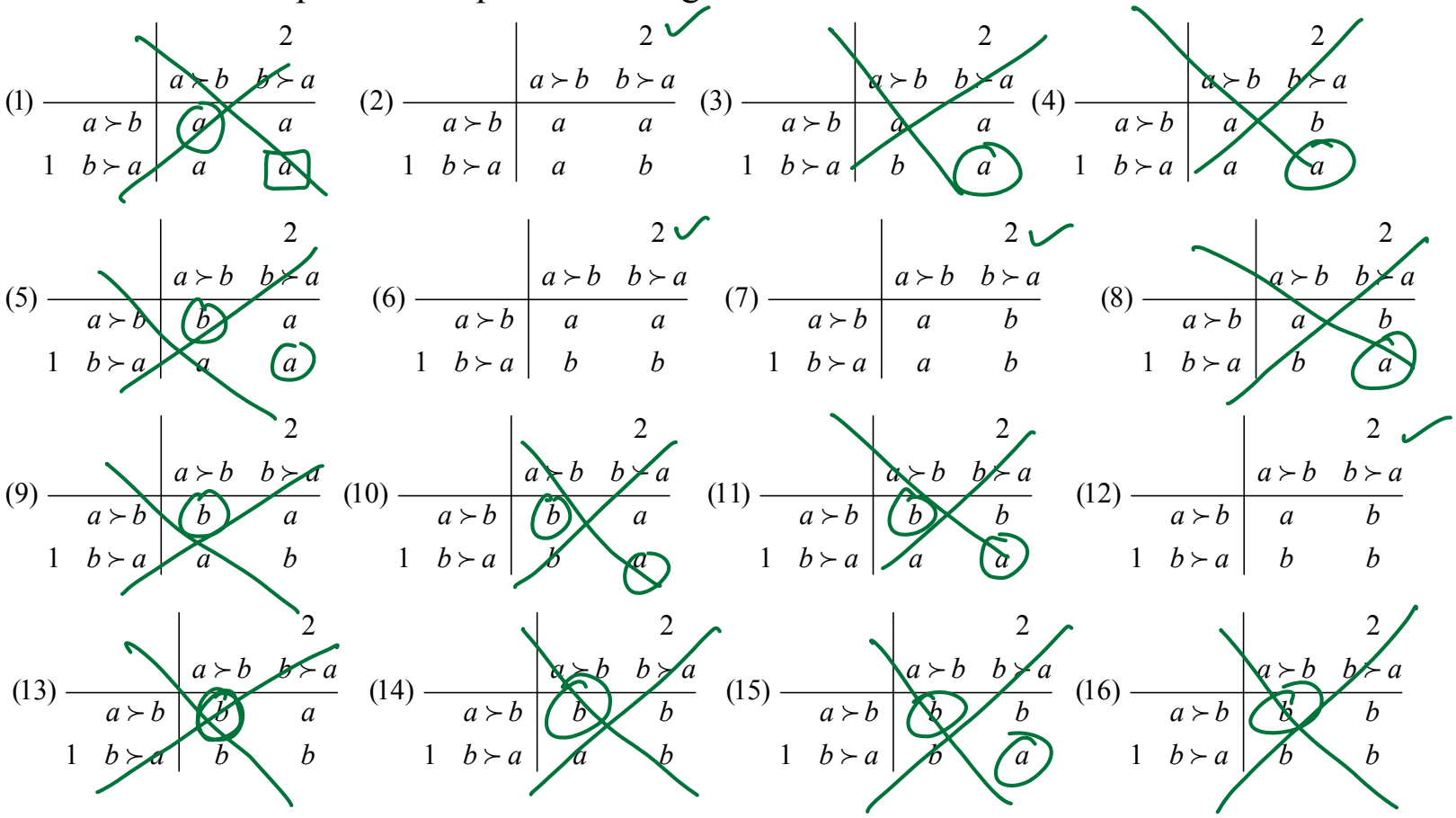
Social Choice Function

Two voters, two alternatives:



$2^4 = 16$

First requirement: UNANIMITY. A good SCF should be such that if both voters put the same alternative at the top of their reported ranking then that alternative should be chosen.



By imposing unanimity we are left with:

1 is a dictator

(2)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	a	b

(6)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	b	b

2 is a dictator

(7)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	a	b

(12)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	b	b

Second requirement: NON-DICTATORSHIP. A good SCF should be such that there is no individual whose top alternative is always chosen, that is, if he reports $a \succ b$ then a is chosen and if he reports $b \succ a$ then b is chosen.

By imposing **Unanimity** and **Non-Dictatorship** we are left with

(2)			2	
		$a \succ b$	$b \succ a$	
	$a \succ b$	a	a	(a is chosen, except when both rank b at the top)
1	$b \succ a$	a	b	

(12)			2	
		$a \succ b$	$b \succ a$	
	$a \succ b$	a	b	(b is chosen, except when both rank a at the top)
1	$b \succ a$	b	b	

Third requirement: NON-MANIPULABILITY. A good SCF should be such that there is no situation where an individual can gain by reporting a false ranking (that is, a ranking which is not her true ranking). Both of the remaining two rankings satisfy this requirement.

Now two voters but three alternatives: *a, b, c*.

$c \succ_2 a \succ_2 b$

2's ranking → / 1's ranking ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	a	a	b	b	c	c
acb	a	a	b	b	c	c
bac	a	a	b	b	c	c
bca	a	a	b	b	c	c
cab	a	a	b	b	c	c
cba	a	a	b	b	c	c

$b \succ_1 a \succ_1 c$ ⟶ *bac*

by Unanimity

to make 2 a dictator add the green outputs

1's prediction of 2's vote

Sufficient to show that 2 is not a dictator

Unanimity ✓

2's ranking → / 1's ranking ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

$b \succ_1 c \succ_1 a$

true ranking of 1

bca

Sufficient to show that 1 is not a dictator

cba lie

Does it satisfy **Unanimity**? Yes

Nondictatorship? Yes

Satisfies **Unanimity** and **Non-Dictatorship**, but **fails Non-Manipulability:**

2's ranking ➔ / 1's ranking ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Gibbard-Satterthwaite theorem: If there are at least 3 alternatives then any SCF which satisfies unanimity and non-dictatorship must be manipulable.