Social Preference Function input: $(\succsim_1, \succsim_2, \dots, \succsim_n)$

output: $\succsim$ society's ranking

Arrow's axioms

- **Axiom 1: Unrestricted Domain or Freedom of Expression**

  At the individual level, any complete and transitive ranking should be allowed.

- **Axiom 2: Rationality**

  Also the social ranking should be complete and transitive

- **Axiom 3: Unanimity or Pareto Principle**

If, for every $i \in \{1, 2, \dots, n\}$, $x \succ_i y$ then $x \succ y$

Example 1

	1's ranking	2's ranking	3's ranking
best	A	C	B
	B	A	C
worst	C	B	A

A versus B no unanimous ranking

A versus C " "

B versus C " "

Unanimity principle imposes no restrictions

Example 2

	1's ranking	2's ranking	3's ranking
best	A	C	A, B
	B	A	
worst	C	B	C

A v. B : no unanimous ranking

A v. C : "

B v C : "

Unanimity principle imposes no restrictions

Example 3

	1's ranking	2's ranking	3's ranking
best	A	C	A
	B	A	C
worst	C	B	B

$A \succ_1 B$, $A \succ_2 B$, $A \succ_3 B$

unanimous then

unanimity requires $A \succ B$

for A and C and for B and C unanimity imposes no restictions

- **Axiom 4: Non-dictatorship**

There is no dictator, that is, there is no individual i such that, for every two alternatives x and y, if $x \succ_i y$ then $x \succ y$.

Equivalently: for every $i \in \{1, 2, \ldots, n\}$ there exists at least one pair (x, y) such that

$x \succ_i y$ but $y \succsim x$

• Axiom 5: Independence of Irrelevant Alternatives

The social ranking of x and y should depend only on how the individuals rank x and y

(1)

	individual 1	individual 2
best	A	A, B
	B	
worst	C	C

We are trying to rank A and B for society

	1	2
best	A	A, B, C
	B	
worst	C	

	1	2
best	A	C
	B	
worst	C	A, B

In all these rankings society's ranking of A and B must be the same

	1	2
best	C	A, B
	A	
worst	B	C

	1	2
best	A, C	A, B
worst	B	C

	1	2
best	A	A, B
	C	
worst	B	C

	1	2
best	A	A, B
worst	B, C	C

	1	2
best	C	A,B,C
	A	
worst	B	

	1	2
best	A, C	A,B,C
worst	B	

	1	2
best	A	A,B,C
	C	
worst	B	

	1	2
best	A	A,B,C
worst	B, C	

	1	2
best	C	C
	A	
worst	B	A,B

	1	2
best	A, C	C
worst	B	A,B

	1	2
best	A	C
	C	
worst	B	A,B

	1	2
best	A	C
worst	B, C	A,B

If there are only two alternatives the Independence of Irrelevant Alternatives axiom is trivially satisfied.

Remark 1. *If there are only two alternatives (and any number of individuals) then the method of majority voting satisfies all of Arrow's axioms.*

Arrow's Impossibility Theorem

If the number of alternatives is at least three,

there is no social preference function that satisfies the five axioms.

equivalently: if you find a SPF that satisfies four of the axioms, then it must violate the fifth axiom.

Arrow's axioms

Unrestricted Domain or Freedom of Expression	FE	
Rationality	**R**	completeness transitivity
Unanimity or Pareto	**U**	
Non-Dictatorship	**ND**	
Independence of Irrelevant Alternatives	**IIA**	

Majority Rule with 2 alternatives

Plurality Rule with 2 alternatives

These two satisfy all of Arrow's axioms

Majority Rule with more than 2 alternatives

Completeness yes Transitivity fails

U yes

ND yes

IIA yes

FE yes

Plurality Rule with more than 2 alternatives *n voters*

majority =

if n is even: number or individuals $\geq \frac{n}{2}+1$

if n is odd: number or individuals $\geq \frac{n+1}{2}$

Majority rule: **if a majority prefers x to y then society prefers x to y**
if a majority prefers y to x then society prefers y to x
otherwise society is indifferent between x and y

Plurality rule: **if the number of individuals who prefer x to y is grater than the number of individuals who prefer y to x then society prefers x to y**

if the number of individuals who prefer y to x is grater than the number of individuals who prefer x to y then society prefers y to x

otherwise society is indifferent between x and y