Arrow's Impossibility Theorem

If the number of alternatives is at least three,
there is no social preference function that satisfies the five axioms.

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Borda count

- n alternatives, m voters
- each voter submits a *strict* ranking of the alternatives
- for each voter the top alternative receives n points, the second $(n-1)$ points, etc.
- for each alternative we take the sum of each individual score
- alternatives are ranked according to the computed score

$n = 3$
$m = 3$

	Voter 1	Voter 2	Voter 3	score
best	a	b	c	3
	b	a	b	2
worst	c	c	a	1

a : 3 + 2 + 1 = 6
b : 2 + 3 + 2 = 7
c : 1 + 1 + 3 = 5

best b
a
worst c

Social ranking:

Which of Arrow's axioms does the Borda count satisfy?

1. Unrestricted domain? Yes

or Freedom of Expression

2. Rationality? complete / transitive Yes

3. Unanimity? Yes

4. Non-dictatorship? Yes

5. IIA must be violated by Arrow's theorem

5. Independence of irrelevant alternatives?

$n = 4$
$m = 7$

Voter:	1	2	3	4	5	6	7	
best	x	a	b	x	a	b	x	4
	c	x	a	c	x	a	c	3
	b	c	x	b	c	x	b	2
worst	a	b	c	a	b	c	a	1

$x: 4+3+2+4+3+2+4 = 22$
$c: 15$
$b: 16$
$a: 17 = 1+4+3+1+4+3+1$

Social ranking:

best x
a
b
worst c

Focus on $a > b$ for society

Voter:	1	2	3	4	5	6	7	Score
best	c	a	b	c	a	b	c	4
	b	c	a	b	c	a	b	3
	a	b	c	a	b	c	a	2
worst	x	x	x	x	x	x	x	1

by IIA we need still $a > b$

total scores:
$c : 22$
$b : 21$
$a : 20$
$x : 7$

$\rightsquigarrow$

Social ranking:

best c
b
a
worst x

$b > a$

Kemeny-Young method

For each pair of alternatives, x and y, count:
(1) the number of individuals for whom $x \succ y$; denote it by $\#(x \succ y)$,
(2) the number of individuals for whom $x \sim y$; denote it by $\#(x \sim y)$,
(3) the number of individuals from whom $y \succ x$ denote it by $\#(y \succ x)$.

Next go through all the complete and transitive rankings of X and for each compute a total score by adding up the scores of each pairwise ranking.

Example: $X = \{A, B, C\}$, $S = \{1, 2, 3, 4, 5\}$

	voter 1	voter 2	voter 3	voter 4	voter 5
best	A	B	B	C	B
	B	C	C	A	A
worst	C	A	A	B	C

input

how close is this to the input?

Ranking	Score	
$A \succ B \succ C$	$\#(A>B)=2$, $\#(B>C)=4$, $\#(A>C)=2$	$2+4+2=8$
$A \succ C \succ B$	$\#(A>B)=2$, $\#(C>B)=1$, $\#(A>C)=2$	$2+1+2=5$
$B \succ A \succ C$	$\#(B>A)=3$, $\#(B>C)=4$, $\#(A>C)=2$	$3+4+2=9$
$B \succ C \succ A$	$\#(B>C)=4$, $\#(C>A)=3$, $\#(B>A)=3$	$4+3+3=10$
$C \succ A \succ B$	$\#(C>A)=3$, $\#(C>B)=1$, $\#(A>B)=2$	$3+1+2=6$
$C \succ B \succ A$	$\#(C>B)=1$, $\#(B>A)=3$, $\#(C>A)=3$	$1+3+3=7$

Social ranking: $B \succ C \succ A$

Which of Arrow's axioms does Kemeny-Young satisfy?

1. Unrestricted domain? Yes

or Freedom of expression

2. Rationality? completeness, transitivity — Yes

3. Unanimity?
requires some proof: see textbook

Yes

4. Non-dictatorship? Yes

By Arrow's axiom IIA must be violated

5. Independence of irrelevant alternatives?

	1	2	3	4	5	6	7
best	A	A	A	B	B	C	C
	B	B	B	C	C	A	A
worst	C	C	C	A	A	B	B

Ranking	Score
$A \succ B \succ C$	13
$A \succ C \succ B$	10
$B \succ A \succ C$	10
$B \succ C \succ A$	11
$C \succ A \succ B$	11
$C \succ B \succ A$	8 $= \underbrace{\#(C > B)}_{=2} + \underbrace{\#(B > A)}_{=2} + \underbrace{\#(C > A)}_{=4}$

Social ranking:

A > C

	1	2	3	**4**	**5**	6	7
best	A	A	A	**C**	**C**	C	C
	B	B	B	**B**	**B**	A	A
worst	C	C	C	**A**	**A**	B	B

Ranking	Score
$A \succ B \succ C$	11
$A \succ C \succ B$	12
$B \succ A \succ C$	8
$B \succ C \succ A$	9
$C \succ A \succ B$	13
$C \succ B \succ A$	10

Social ranking:

C > A