

PRACTICE PROBLEMS on cooperative games

The answers are at the end of this file starting from page 5

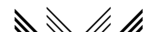
VERY IMPORTANT: do **not** look at the answers until you have made a VERY serious effort to solve the problem. If you turn to the answers to get clues or help, you are wasting a chance to test how well you are prepared for the exams. I will **not** give you more practice problems later on.



1. Consider the following cooperative game: $N = \{1, 2, 3\}$ and

$$\begin{aligned}v(\{1\}) &= 10, & v(\{2\}) &= 6, & v(\{3\}) &= 8 \\v(\{1,2\}) &= 18, & v(\{1,3\}) &= 24, & v(\{2,3\}) &= 16 \\v(\{1,2,3\}) &= 30.\end{aligned}$$

Find the core.



2. Consider the following cooperative game: $N = \{1, 2, 3\}$ and

$$\begin{aligned}v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \\v(\{1,2\}) &= 40, & v(\{1,3\}) &= 0, & v(\{2,3\}) &= 50 \\v(\{1,2,3\}) &= 50\end{aligned}$$

Find the core.



3. Consider the following cooperative game: $N = \{1, 2\}$ and

$$v(\{1\}) = 2, \quad v(\{2\}) = 5, \quad v(\{1,2\}) = 8.$$

(a) Find the core.

(b) If imputations are required to be integer-valued (that is, the amount given to each player is an integer), list all the imputations in the core.

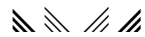


4. Consider the following cooperative game: $N = \{1, 2, 3\}$ and

$$\begin{aligned}v(\{1\}) &= 4, & v(\{2\}) &= 6, & v(\{3\}) &= 3 \\v(\{1,2\}) &= 14, & v(\{1,3\}) &= 12, & v(\{2,3\}) &= 16 \\v(\{1,2,3\}) &= 18\end{aligned}$$

For each of the following imputations (x_1, x_2, x_3) determine if it is in the core:

1. (6, 6, 6)
2. (4, 6, 8)
3. (7, 7, 4)
4. (8, 8, 2)



5. Consider the following cooperative game: $N = \{1, 2, 3\}$ and

$$\begin{aligned}v(\{1\}) &= 2, & v(\{2\}) &= 4, & v(\{3\}) &= 1 \\v(\{1,2\}) &= 12, & v(\{1,3\}) &= 10, & v(\{2,3\}) &= 14 \\v(\{1,2,3\}) &= 16\end{aligned}$$

Prove that the core is empty.



6. Consider the following cooperative game: $N = \{1, 2, 3, 4\}$ and

$$\begin{aligned}v(\{1\}) &= v(\{2\}) = 2, & v(\{3\}) &= v(\{4\}) = 4 \\v(\{1,2\}) &= v(\{1,3\}) = v(\{1,4\}) = 6, & v(\{2,3\}) &= 9, & v(\{3,4\}) &= 10, \\v(\{1,2,3\}) &= v(\{1,2,4\}) = v(\{2,3,4\}) = 13, \\v(\{1,2,3,4\}) &= 18\end{aligned}$$

For each of the following imputations (x_1, x_2, x_3, x_4) determine if it is in the core:

1. (4, 4, 5, 5)
2. (2, 4, 6, 6)
3. (4, 5, 5, 4)



7. Consider the following cooperative game: $N = \{1, 2, 3, 4\}$ and

$$v(\{1\}) = v(\{2\}) = 4, \quad v(\{3\}) = v(\{4\}) = 6$$

$$v(\{1,2\}) = v(\{1,3\}) = v(\{1,4\}) = 8, \quad v(\{2,3\}) = 10, \quad v(\{2,4\}) = 10, \quad v(\{3,4\}) = 12,$$

$$v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{2,3,4\}) = 14,$$

$$v(\{1,2,3,4\}) = 18$$

Is the core non-empty?



8. Consider the following cooperative game: $N = \{1, 2, 3\}$ and

$$v(\{1\}) = 10, \quad v(\{2\}) = 8, \quad v(\{3\}) = 6$$

$$v(\{1,2\}) = 24, \quad v(\{1,3\}) = 22, \quad v(\{2,3\}) = 18$$

$$v(\{1,2,3\}) = 34$$

Find the Shapley value.



9. Consider the following cooperative game: $N = \{1, 2, 3\}$ and

$$v(\{1\}) = 80, \quad v(\{2\}) = 60, \quad v(\{3\}) = 30$$

$$v(\{1,2\}) = 180, \quad v(\{1,3\}) = 160, \quad v(\{2,3\}) = 120$$

$$v(\{1,2,3\}) = 260.$$

Find the Shapley value



10. Consider again the game of Exercise 9. Is Player 1 a dummy player?



11. Consider again the game of Exercise 9. Are Players 1 and 2 interchangeable?



12. Consider the following cooperative game: $N = \{1, 2, 3\}$ and

$$\begin{aligned}v(\{1\}) &= 2, & v(\{2\}) &= 4, & v(\{3\}) &= 2 \\v(\{1,2\}) &= 8, & v(\{1,3\}) &= 10, & v(\{2,3\}) &= 8 \\v(\{1,2,3\}) &= 12\end{aligned}$$

(a) Are Players 1 and 3 interchangeable?

(b) Find the Shapley value.

(c) Is the Shapley value in the core?



13. Consider the following cooperative game: $N = \{1, 2, 3\}$ and

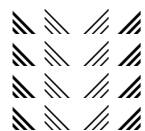
$$\begin{aligned}v(\{1\}) &= 2, & v(\{2\}) &= 4, & v(\{3\}) &= 6 \\v(\{1,2\}) &= 6, & v(\{1,3\}) &= 8, & v(\{2,3\}) &= 12 \\v(\{1,2,3\}) &= 14\end{aligned}$$

(a) Are any two players interchangeable?

(b) Is any player a dummy player?

(c) Find the Shapley value.

(d) Is the Shapley value in the core?



ANSWERS

1. The core is the set of (x_1, x_2, x_3) such that

$x_1 \geq v(\{1\}) = 10$	(1)
$x_2 \geq v(\{2\}) = 6$	(2)
$x_3 \geq v(\{3\}) = 8$	(3)
$x_1 + x_2 \geq v(\{1,2\}) = 18$	(4)
$x_1 + x_3 \geq v(\{1,3\}) = 24$	(5)
$x_2 + x_3 \geq v(\{2,3\}) = 16$	(6)
$x_1 + x_2 + x_3 = v(\{1,2,3\}) = 30$	(7)

From (5) and (7) we get that $x_2 \leq 6$. This, together with (2), gives

$$x_2 = 6. \quad (8)$$

From (7) and (8) we get that $x_1 + x_3 = 24$ so that

$$x_3 = 24 - x_1. \quad (9)$$

From (4) and (8) we get that

$$x_1 \geq 12. \quad (10).$$

From (6) and (8) we get that $x_3 \geq 10$ and this, together with (9) gives $x_1 \leq 14$.

Thus the core is the set of triples (x_1, x_2, x_3) such that $12 \leq x_1 \leq 14$, $x_2 = 6$ and $x_3 = 24 - x_1$.

2. The core is the set of (x_1, x_2, x_3) such that

$x_1 \geq v(\{1\}) = 0$	(1)
$x_2 \geq v(\{2\}) = 0$	(2)
$x_3 \geq v(\{3\}) = 0$	(3)
$x_1 + x_2 \geq v(\{1,2\}) = 40$	(4)
$x_1 + x_3 \geq v(\{1,3\}) = 0$	(5)
$x_2 + x_3 \geq v(\{2,3\}) = 50$	(6)
$x_1 + x_2 + x_3 = v(\{1,2,3\}) = 50$	(7)

From (6) and (7) we get that $x_1 \leq 0$. This, together with (1), gives

$$x_1 = 0. \quad (8)$$

From (7) and (8) we get that $x_2 + x_3 = 50$ so that

$$x_3 = 50 - x_2. \quad (9)$$

From (4) and (8) we get that

$$x_2 \geq 40. \quad (10)$$

Thus the core is the set of triples (x_1, x_2, x_3) such that $x_1 = 0$, $x_2 \geq 40$ and $x_3 = 50 - x_2$.

3. (a) The core is the set of (x_1, x_2) such that $x_1 \geq 2$, $x_2 \geq 5$ and $x_1 + x_2 = 8$. Thus the set of pairs $(x_1, 8 - x_1)$ such that $2 \leq x_1 \leq 3$.

(b) Only two: (2, 6) and (3, 5).

4.

1. (6, 6, 6) is not in the core because, for example, the coalition {1,2} can block it with (7,7,0).
2. (4, 6, 8) is not in the core because, for example, the coalition {1,2} can block it with (7,7,0).
3. (7, 7, 4) is not in the core because, for example, the coalition {2,3} can block it with (0,8,8).
4. (8, 8, 2) is not in the core because, for example, the coalition {2,3} can block it with (0,9,7).

5. If (x_1, x_2, x_3) is in the core it must satisfy the following inequalities:

(1) $x_1 + x_2 \geq 12$, (2) $x_1 + x_3 \geq 10$, (3) $x_2 + x_3 \geq 14$

Adding these inequalities we get $2x_1 + 2x_2 + 2x_3 \geq 36$, that is, $x_1 + x_2 + x_3 \geq 18$ which is impossible since $v(\{1,2,3\}) = 16$.

6. 1. $(4, 4, 5, 5)$ is in the core (it satisfies all the inequalities).
 2. $(2, 4, 6, 6)$ is not in the core because the coalition $\{1,2,3\}$ can block it with, for example, $(2.4, 4.3, 6.3, 0)$.
 3. $(4, 5, 5, 4)$ is not in the core because the coalition $\{3,4\}$ can block it with, for example, $(0, 0, 5.5, 4.5)$.

7. No, the core is empty because, in order to be in the core, an imputation (x_1, x_2, x_3, x_4) must be such that $x_3 + x_4 \geq 12$ (otherwise it can be blocked by the coalition $\{3,4\}$) and, furthermore, it must be such that $x_1 \geq 4$ (otherwise it can be blocked by the coalition $\{1\}$) and $x_2 \geq 4$ (otherwise it can be blocked by the coalition $\{2\}$), so that $x_1 + x_2 + x_3 + x_4 \geq 20$, which is impossible, since $v(N) = 18$.

8. The Shapley value is $x_1 = 14$, $x_2 = 11$, $x_3 = 9$ and is calculated as follows:

SHAPLEY VALUE FOR CAPITALIST-WORKERS							
$v(\{1\})$	$v(\{2\})$	$v(\{3\})$	$v(\{1,2\})$	$v(\{1,3\})$	$v(\{2,3\})$	$v(\{1,2,3\})$	
10	8	6	24	22	18	34	
order	probability	player 1's marginal contribution	player 2's marginal contribution	player 3's marginal contribution			
123	1/6	10	14	10			
132	1/6	10	12	12			
213	1/6	16	8	10			
231	1/6	16	8	10			
312	1/6	16	12	6			
321	1/6	16	12	6			
	sum	84	66	54			
					check sum		
	Shapley value	14	11	9	34		

9. The Shapley value is $x_1 = 115$, $x_2 = 85$, $x_3 = 60$ and is calculated as follows:

$v(\{1\})$	$v(\{2\})$	$v(\{3\})$	$v(\{1,2\})$	$v(\{1,3\})$	$v(\{2,3\})$	$v(\{1,2,3\})$
80	60	30	180	160	120	260
order	probability	player 1's marginal contribution	player 2's marginal contribution	player 3's marginal contribution		
123	1/6	80	100	80		
132	1/6	80	100	80		
213	1/6	120	60	80		
231	1/6	140	60	60		
312	1/6	130	100	30		
321	1/6	140	90	30		
	sum	690	510	360		
					check sum	
	Shapley value	115	85	60	260	

- 10.** Player 1 is not a dummy player, because $v(\{1,2\}) - v(\{2\}) = 180 - 60 = 120 > v(\{1\}) = 80$.
- 11.** Players 1 and 2 are not interchangeable because $v(\{1\}) \neq v(\{2\})$.
- 12.** (a) Players 1 and 3 are interchangeable because $v(\{1\}) = v(\{3\})$ and $v(\{1,2\}) - v(\{2\}) = v(\{2,3\}) - v(\{3\}) = 4$.
- (b) The Shapley value is (4, 4, 4).
- (c) The Shapley value is not in the core because (4, 4, 4) can be blocked by the coalition {1,3} with, for example, (5, 0, 5)
- 13.** (a) No two players are interchangeable because $v(\{i\}) \neq v(\{j\})$ for any $i \neq j$.
- (b) Player 1 is a dummy player because $v(\{1,2\}) = v(\{2\}) + v(\{1\})$, $v(\{1,3\}) = v(\{3\}) + v(\{1\})$ and $v(\{1,2,3\}) = v(\{2,3\}) + v(\{1\})$.
- (c) The Shapley value is (2, 5, 7).
- (d) The Shapley value is in the core because it satisfies all the inequalities that define the core.