The answers are at the end of this file starting from page 5

VERY IMPORTANT: do not look at the answers until you have made a VERY serious effort to solve the problem. If you turn to the answers to get clues or help, you are wasting a chance to test how well you are prepared for the exams. I will not give you more practice problems later on.

1. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
\[
\begin{align*}
v(\{1\}) &= 10, & v(\{2\}) &= 6, & v(\{3\}) &= 8 \\
v(\{1, 2\}) &= 18, & v(\{1, 3\}) &= 24, & v(\{2, 3\}) &= 16 \\
v(\{1, 2, 3\}) &= 30.
\end{align*}
\]
Find the core.

2. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
\[
\begin{align*}
v(\{1\}) &= v(\{2\}) = v(\{3\}) &= 0 \\
v(\{1, 2\}) &= 40, & v(\{1, 3\}) &= 0, & v(\{2, 3\}) &= 50 \\
v(\{1, 2, 3\}) &= 50
\end{align*}
\]
Find the core.

3. Consider the following cooperative game: \( N = \{1, 2\} \) and
\[
\begin{align*}
v(\{1\}) &= 2, & v(\{2\}) &= 5, & v(\{1, 2\}) &= 8.
\end{align*}
\]
(a) Find the core.

(b) If imputations are required to be integer-valued (that is, the amount given to each player is an integer), list all the imputations in the core.
4. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and 
\[
\begin{align*}
v(\{1\}) &= 4, & v(\{2\}) &= 6, & v(\{3\}) &= 3 \\
v(\{1,2\}) &= 14, & v(\{1,3\}) &= 12, & v(\{2,3\}) &= 16 \\
v(\{1,2,3\}) &= 18
\end{align*}
\]
For each of the following imputations \((x_1, x_2, x_3)\) determine if it is in the core:
1. \((6, 6, 6)\)
2. \((4, 6, 8)\)
3. \((7, 7, 4)\)
4. \((8, 8, 2)\)

5. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and 
\[
\begin{align*}
v(\{1\}) &= 2, & v(\{2\}) &= 4, & v(\{3\}) &= 1 \\
v(\{1,2\}) &= 12, & v(\{1,3\}) &= 10, & v(\{2,3\}) &= 14 \\
v(\{1,2,3\}) &= 16
\end{align*}
\]
Prove that the core is empty.

6. Consider the following cooperative game: \( N = \{1, 2, 3, 4\} \) and 
\[
\begin{align*}
v(\{1\}) &= v(\{2\}) = v(\{3\}) = v(\{4\}) = 2 \\
v(\{1,2\}) &= v(\{1,3\}) = v(\{1,4\}) = 6, & v(\{2,3\}) &= 9, & v(\{3,4\}) &= 10, \\
v(\{1,2,3\}) &= v(\{1,2,4\}) = v(\{2,3,4\}) = 13, \\
v(\{1,2,3,4\}) &= 18
\end{align*}
\]
For each of the following imputations \((x_1, x_2, x_3, x_4)\) determine if it is in the core:
1. \((4, 4, 5, 5)\)
2. \((2, 4, 6, 6)\)
3. \((4, 5, 5, 4)\)
7. Consider the following cooperative game: \( N = \{1, 2, 3, 4\} \) and
\[
\begin{align*}
v(\{1\}) &= v(\{2\}) = 4, \\v(\{3\}) &= v(\{4\}) = 6 \\
v(\{1,2\}) &= v(\{1,3\}) = v(\{1,4\}) = 8, \\v(\{2,3\}) &= 10, \\v(\{2,4\}) &= 10, \\v(\{3,4\}) &= 12, \\
v(\{1,2,3\}) &= v(\{1,2,4\}) = v(\{2,3,4\}) = 14, \\
v(\{1,2,3,4\}) &= 18
\end{align*}
\]
Is the core non-empty?

8. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
\[
\begin{align*}
v(\{1\}) &= 10, \\v(\{2\}) &= 8, \\v(\{3\}) &= 6 \\
v(\{1,2\}) &= 24, \\v(\{1,3\}) &= 22, \\v(\{2,3\}) &= 18 \\
v(\{1,2,3\}) &= 34
\end{align*}
\]
Find the Shapley value.

9. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
\[
\begin{align*}
v(\{1\}) &= 80, \\v(\{2\}) &= 60, \\v(\{3\}) &= 30 \\
v(\{1,2\}) &= 180, \\v(\{1,3\}) &= 160, \\v(\{2,3\}) &= 120 \\
v(\{1,2,3\}) &= 260
\end{align*}
\]
Find the Shapley value

10. Consider again the game of Exercise 9. Is Player 1 a dummy player?

11. Consider again the game of Exercise 9. Are Players 1 and 2 interchangeable?
12. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and

\[
\begin{align*}
v(\{1\}) &= 2, & v(\{2\}) &= 4, & v(\{3\}) &= 2 \\
v(\{1,2\}) &= 8, & v(\{1,3\}) &= 10, & v(\{2,3\}) &= 8 \\
v(\{1,2,3\}) &= 12
\end{align*}
\]

(a) Are Players 1 and 3 interchangeable?
(b) Find the Shapley value.
(c) Is the Shapley value in the core?

13. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and

\[
\begin{align*}
v(\{1\}) &= 2, & v(\{2\}) &= 4, & v(\{3\}) &= 6 \\
v(\{1,2\}) &= 6, & v(\{1,3\}) &= 8, & v(\{2,3\}) &= 12 \\
v(\{1,2,3\}) &= 14
\end{align*}
\]

(a) Are any two players interchangeable?
(b) Is any player a dummy player?
(c) Find the Shapley value.
(d) Is the Shapley value in the core?
1. The core is the set of \((x_1, x_2, x_3)\) such that

| \(x_1 \geq v(\{1\}) = 10\) | (1) |
| \(x_2 \geq v(\{2\}) = 6\) | (2) |
| \(x_3 \geq v(\{3\}) = 8\) | (3) |
| \(x_1 + x_2 \geq v(\{1,2\}) = 18\) | (4) |
| \(x_1 + x_3 \geq v(\{1,3\}) = 24\) | (5) |
| \(x_2 + x_3 \geq v(\{2,3\}) = 16\) | (6) |
| \(x_1 + x_2 + x_3 = v(\{1,2,3\}) = 30\) | (7) |

From (5) and (7) we get that \(x_2 \leq 6\). This, together with (2), gives \(x_2 = 6\). \(\text{(8)}\)

From (7) and (8) we get that \(x_1 + x_3 = 24\) so that \(x_3 = 24 - x_1\). \(\text{(9)}\)

From (4) and (8) we get that \(x_1 \geq 12\). \(\text{(10)}\)

From (6) and (8) we get that \(x_3 \geq 10\) and this, together with (9) gives \(x_1 \leq 14\).

Thus the core is the set of triples \((x_1, x_2, x_3)\) such that \(12 \leq x_1 \leq 14, x_2 = 6\) and \(x_3 = 24 - x_1\).
2. The core is the set of \((x_1, x_2, x_3)\) such that

<table>
<thead>
<tr>
<th>(x_1 \geq v({1}) = 0)</th>
<th>(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2 \geq v({2}) = 0)</td>
<td>(2)</td>
</tr>
<tr>
<td>(x_3 \geq v({3}) = 0)</td>
<td>(3)</td>
</tr>
<tr>
<td>(x_1 + x_2 \geq v({1,2}) = 40)</td>
<td>(4)</td>
</tr>
<tr>
<td>(x_1 + x_3 \geq v({1,3}) = 0)</td>
<td>(5)</td>
</tr>
<tr>
<td>(x_2 + x_3 \geq v({2,3}) = 50)</td>
<td>(6)</td>
</tr>
<tr>
<td>(x_1 + x_2 + x_3 = v({1,2,3}) = 50)</td>
<td>(7)</td>
</tr>
</tbody>
</table>

From (6) and (7) we get that \(x_1 \leq 0\). This, together with (1), gives

\[ x_1 = 0. \quad (8) \]

From (7) and (8) we get that \(x_2 + x_3 = 50\) so that

\[ x_3 = 50 - x_2. \quad (9) \]

From (4) and (8) we get that

\[ x_2 \geq 40. \quad (10) \]

Thus the core is the set of triples \((x_1, x_2, x_3)\) such that \(x_1 = 0\), \(x_2 \geq 40\) and \(x_3 = 50 - x_2\).

3. (a) The core is the set of \((x_1, x_2)\) such that \(x_1 \geq 2\), \(x_2 \geq 5\) and \(x_1 + x_2 = 8\). Thus the set of pairs \((x_1, 8 - x_1)\) such that \(2 \leq x_1 \leq 3\).

(b) Only two: (2, 6) and (3,5).

4. 1. \((6, 6, 6)\) is not in the core because, for example, the coalition \(\{1,2\}\) can block it with \((7,7,0)\).
2. \((4, 6, 8)\) is not in the core because, for example, the coalition \(\{1,2\}\) can block it with \((7,7,0)\).
3. \((7, 7, 4)\) is not in the core because, for example, the coalition \(\{2,3\}\) can block it with \((0,8,8)\).
4. \((8, 8, 2)\) is not in the core because, for example, the coalition \(\{2,3\}\) can block it with \((0,9,7)\).
5. If \((x_1, x_2, x_3)\) is in the core it must satisfy the following inequalities:

\[
\begin{align*}
(1) \quad & x_1 + x_2 \geq 12, \\
(2) \quad & x_1 + x_3 \geq 10, \\
(3) \quad & x_2 + x_3 \geq 14
\end{align*}
\]

Adding these inequalities we get \(2x_1 + 2x_2 + 2x_3 \geq 36\), that is, \(x_1 + x_2 + x_3 \geq 18\) which is impossible since \(v(\{1,2,3\}) = 16\).

6. 1. \((4, 4, 5, 5)\) is in the core (it satisfies all the inequalities).

2. \((2, 4, 6, 6)\) is not in the core because the coalition \(\{1,2,3\}\) can block it with, for example, \((2.4, 4.3, 6.3, 0)\).

3. \((4, 5, 5, 4)\) is not in the core because the coalition \(\{3,4\}\) can block it with, for example, \((0, 0, 5.5, 4.5)\).

7. No, the core is empty because, in order to be in the core, an imputation \((x_1, x_2, x_3, x_4)\) must be such that \(x_3 + x_4 \geq 12\) (otherwise it can be blocked by the coalition \(\{3,4\}\)) and, furthermore, it must be such that \(x_i \geq 4\) (otherwise it can be blocked by the coalition \(\{1\}\)) and \(x_i \geq 4\) (otherwise it can be blocked by the coalition \(\{2\}\)), so that \(x_1 + x_2 + x_3 + x_4 \geq 20\), which is impossible, since \(v(N) = 18\).

8. The Shapley value is \(x_1 = 14, x_2 = 11, x_2 = 9\) and is calculated as follows:

<table>
<thead>
<tr>
<th>(v((1)))</th>
<th>(v((2)))</th>
<th>(v((3)))</th>
<th>(v((1,2)))</th>
<th>(v((1,3)))</th>
<th>(v((2,3)))</th>
<th>(v((1,2,3)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8</td>
<td>6</td>
<td>24</td>
<td>22</td>
<td>18</td>
<td>34</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>order</th>
<th>probability</th>
<th>player 1's marginal contribution</th>
<th>player 2's marginal contribution</th>
<th>player 3's marginal contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>1/6</td>
<td>10</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>132</td>
<td>1/6</td>
<td>10</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>213</td>
<td>1/6</td>
<td>16</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>231</td>
<td>1/6</td>
<td>16</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>312</td>
<td>1/6</td>
<td>16</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>321</td>
<td>1/6</td>
<td>16</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>sum</td>
<td></td>
<td>84</td>
<td>66</td>
<td>54</td>
</tr>
</tbody>
</table>

| Shapley value | 14 | 11 | 9 | 34 |

9. The Shapley value is \(x_1 = 115, x_2 = 85, x_3 = 60\) and is calculated as follows:
<table>
<thead>
<tr>
<th>order</th>
<th>probability</th>
<th>player 1's marginal contribution</th>
<th>player 2's marginal contribution</th>
<th>player 3's marginal contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>1/6</td>
<td>80</td>
<td>100</td>
<td>80</td>
</tr>
<tr>
<td>132</td>
<td>1/6</td>
<td>80</td>
<td>100</td>
<td>80</td>
</tr>
<tr>
<td>213</td>
<td>1/6</td>
<td>120</td>
<td>60</td>
<td>80</td>
</tr>
<tr>
<td>231</td>
<td>1/6</td>
<td>140</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>312</td>
<td>1/6</td>
<td>130</td>
<td>100</td>
<td>30</td>
</tr>
<tr>
<td>321</td>
<td>1/6</td>
<td>140</td>
<td>90</td>
<td>30</td>
</tr>
<tr>
<td>sum</td>
<td></td>
<td>690</td>
<td>510</td>
<td>360</td>
</tr>
<tr>
<td>Shapley value</td>
<td>115</td>
<td>85</td>
<td>60</td>
<td>260</td>
</tr>
</tbody>
</table>

10. Player 1 is not a dummy player, because $v(\{1,2\}) - v(\{2\}) = 180 - 60 = 120 > v(\{1\}) = 80$.

11. Players 1 and 2 are not interchangeable because $v(\{1\}) \neq v(\{2\})$.

12. (a) Players 1 and 3 are interchangeable because $v(\{1\}) = v(\{3\})$ and $v(\{1,2\}) - v(\{2\}) = v(\{2,3\}) - v(\{3\}) = 4$.

(b) The Shapley value is (4, 4, 4).

(e) The Shapley value is not in the core because (4, 4, 4) can be blocked by the coalition \{1,3\} with, for example, (5, 0, 5)

13. (a) No two players are interchangeable because $v(\{i\}) \neq v(\{j\})$ for any $i \neq j$.

(b) Player 1 is a dummy player because $v(\{1,2\}) = v(\{2\}) + v(\{1\})$, $v(\{1,3\}) = v(\{3\}) + v(\{1\})$ and $v(\{1,2,3\}) = v(\{2,3\}) + v(\{1\})$.

(c) The Shapley value is (2, 5, 7).

(d) The Shapley value is in the core because it satisfies all the inequalities that define the core.