Inverse demand is given by \( P_H(Q) = \frac{36-Q}{b_H} \) and \( P_L(Q) = \frac{36-Q}{b_L} \).

(a) Let \( W_H(Q) \) be the willingness to pay of the \( H \) consumer for \( Q \) units of output. Then

\[
W_H(Q) = \frac{1}{b_H} \left( 36Q - \frac{Q^2}{2} \right).
\]

For Option 1 the monopolist would choose within the set \( \{(Q,V) : V = W_H(Q)\} \). The corresponding profits are \( \pi_1(Q) = \frac{1}{b_H} \left( 36Q - \frac{Q^2}{2} \right) - 3Q \). This function is strictly concave. Solving \( \frac{d\pi_1}{dQ} = 0 \) we get \( Q_1^* = 36 - 3b_H \), \( V_1^* = \frac{9}{2} \left( \frac{144 - b_H^2}{b_H} \right) \) with corresponding profits \( \pi_1^* = \frac{9}{2} \left( \frac{(12 - b_H)^2}{b_H} \right) \).

(b) Let \( W_L(Q) \) be the willingness to pay of the \( L \) consumer for \( Q \) units of output. Then

\[
W_L(Q) = \frac{1}{b_L} \left( 36Q - \frac{Q^2}{2} \right).
\]

For Option 2 the monopolist would choose within the set \( \{(Q,V) : V = W_L(Q)\} \). The corresponding profits are \( \pi_2(Q) = 2 \left[ \frac{1}{b_L} \left( 36Q - \frac{Q^2}{2} \right) - 3Q \right] \). This function is strictly concave. Solving \( \frac{d\pi_2}{dQ} = 0 \) we get \( Q_2^* = 36 - 3b_L \), \( V_2^* = \frac{9}{2} \left( 144 - \frac{b_L^2}{b_L} \right) \) with corresponding profits \( \pi_2^* = \frac{9}{2} \left( \frac{(12 - \frac{b_L}{b_L})^2}{\frac{b_L}{b_L}} \right) \).

(c) Let \((Q_H,V_H)\) and \((Q_L,V_L)\) be the two packages. Then the incentive compatibility constraints are:

1. \( V_L \leq W_L(Q_L) \) : the \( L \)-consumer is willing to buy “her” package
2. \( W_L(Q_H) - V_L \geq W_L(Q_H) - V_H \) : incentive compatibility constraint for the \( L \)-consumer (she does not prefer the \( H \) package to the \( L \) package)
3. \( V_H \leq W_H(Q_H) \) : the \( H \)-consumer is willing to buy “her” package
4. \( W_H(Q_H) - V_H \geq W_H(Q_H) - V_L \) incentive compatibility constraint for the \( H \)-consumer (she does not prefer the \( L \) package to the \( H \) package);

Since \( W_H(Q) > W_L(Q) \), (3) follows, as a strict inequality, from (1) and (4). Let \( S \) be the set of pairs \( \{(Q_H,V_H),(Q_L,V_L)\} \) that satisfy (1) – (4) and let \( \left( (Q_H^*,V_H^*),(Q_L^*,V_L^*) \right) \in S \) be a pair that
maximizes the firm’s profits within \( S \). Then (1) and (4) must be satisfied as equalities, that is,

\[
V^*_L = W(Q^*_L) = \frac{1}{b_L} \left( 36Q^*_L - \frac{(Q^*_L)^2}{2} \right)
\]

and

\[
V^*_H = W_H(Q_H^*) - W_L(Q_L^*) + V^*_L = \frac{1}{b_H} \left( 36Q_H^* - \frac{(Q_H^*)^2}{2} \right) - 36Q^*_L - \frac{(Q^*_L)^2}{2} + \frac{1}{b_L} \left( 36Q^*_L - \frac{(Q^*_L)^2}{2} \right)
\]

(d) Let \( S^* \) be the set of pairs \( (Q_H^*, V_H^*), (Q_L^*, V_L^*) \) that satisfy the two boxed equalities. Then if \( (Q_H^*, V_H^*), (Q_L^*, V_L^*) \) maximizes the firm’s profits in \( S \) it must belong to \( S^* \). Within \( S^* \) the profit of the firm is given by

\[
\pi = (V_H^* - 3Q_H^*) + (V_L^* - 3Q_L^*)
\]

The Hessian of \( \pi \) is

\[
H = \begin{bmatrix}
\frac{\partial^2 \pi}{\partial Q_H^* \partial Q_H^*} &= -\frac{1}{b_H} \\
\frac{\partial^2 \pi}{\partial Q_L^* \partial Q_H^*} &= 0 \\
\frac{\partial^2 \pi}{\partial Q_L^* \partial Q_L^*} &= 0 \\
\frac{\partial^2 \pi}{\partial (Q_H^*)^2} &= \frac{b_L - 2b_H}{b_L b_H}
\end{bmatrix}
\]

Which is negative definite since \( b_L < 2b_H \). Thus the maximum of \( \pi \) is found by solving

\[
\frac{\partial \pi}{\partial Q_H^*} = 0 \quad \text{and} \quad \frac{\partial \pi}{\partial Q_L^*} = 0
\]

The solution is

\[
Q_H^* = 36 - 3b_H \quad \text{and} \quad Q_L^* = 3 \frac{24b_H - 12b_L - b_L b_H}{2b_H - b_L}
\]

(e) When \( b_H = 3 \) and \( b_L = 4 \) then \( \pi_1^* = 121.5 \), \( \pi_2^* = 144 \) and \( \pi_3^* = 148.5 \). Thus Option 3 is the best.

The two packages are \( (Q_H = 27, V_H = 162) \) and \( (Q_L = 18, V_L = 121.5) \) with implied per-unit prices of \( P_H = 6 \) and \( P_L = 6.75 \). The \( L \)-consumer gets zero surplus, while the \( H \)-consumer gets a surplus of \( W_H(27) - 162 = 202.5 - 162 = 40.5 \). Thus total surplus is

\[
\pi_3^* + 9 = 148.5 + 40.5 = 189
\]