COURNOT DUOPOLY: an example

Let the inverse demand function and the cost function be given by

\[ P = 50 - 2Q \quad \text{and} \quad C = 10 + 2q \]

respectively, where \( Q \) is total industry output and \( q \) is the firm’s output.

First consider first the case of **uniform-pricing monopoly**, as a benchmark. Then in this case \( Q = q \) and the profit function is

\[ \pi(Q) = (50 - 2Q)Q - 10 - 2Q = 48Q - 2Q^2 - 10. \]

Solving \( \frac{d\pi}{dQ} = 0 \) we get \( Q = 12, P = 26, \pi = 278, CS = \frac{12(50-26)}{2} = 144, TS = 278 + 144 = 422. \)

<table>
<thead>
<tr>
<th>MONOPOLY</th>
<th>Q</th>
<th>P</th>
<th>( \pi )</th>
<th>CS</th>
<th>TS</th>
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<tr>
<td>12</td>
<td>26</td>
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Now let us consider the case of two firms, or **duopoly**. Let \( q_1 \) be the output of firm 1 and \( q_2 \) the output of firm 2. Then \( Q = q_1 + q_2 \) and the profit functions are:

\[ \pi_1(q_1,q_2) = q_1 [50 - 2 (q_1 + q_2)] - 10 - 2q_1 \]
\[ \pi_2(q_1,q_2) = q_2 [50 - 2 (q_1 + q_2)] - 10 - 2q_2 \]

A Nash equilibrium is a pair of output levels \( (q_1^*, q_2^*) \) such that:

\[ \pi_1(q_1^*, q_2^*) \geq \pi_1(q_1^*, q_2) \quad \text{for all} \quad q_1 \geq 0 \]

and

\[ \pi_2(q_1^*, q_2^*) \geq \pi_2(q_1^*, q_2) \quad \text{for all} \quad q_2 \geq 0. \]
This means that, fixing $q_2$ at the value $q_2^*$ and considering $\pi_1$ as a function of $q_1$ alone, this function is maximized at $q_1 = q_1^*$. But a necessary condition for this to be true is that 

$$\frac{\partial \pi_1}{\partial q_1}(q_1^*, q_2^*) = 0.$$ 

Similarly, fixing $q_1$ at the value $q_1^*$ and considering $\pi_2$ as a function of $q_2$ alone, this function is maximized at $q_2 = q_2^*$. But a necessary condition for this to be true is that 

$$\frac{\partial \pi_2}{\partial q_2}(q_1^*, q_2^*) = 0.$$ 

Thus the Nash equilibrium is found by solving the following system of two equations in the two unknowns $q_1$ and $q_2$:

$$\begin{align*}
\frac{\partial \pi_1}{\partial q_1}(q_1^*, q_2^*) &= 50 - 4q_1 - 2q_2 - 2 = 0 \\
\frac{\partial \pi_2}{\partial q_2}(q_1^*, q_2^*) &= 50 - 2q_1 - 4q_2 - 2 = 0
\end{align*}$$

The solution is $q_1^* = q_2^* = 8$, $Q = 16$, $P = 18$, $\pi_1 = \pi_2 = 118$, $CS = \frac{16(50-18)}{2} = 256$, $TS = 118 + 118 + 256 = 492$.

Let us compare the two.

<table>
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<tr>
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<table>
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<tr>
<th>DUOPOLY</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>Q</th>
<th>P</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
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Thus competition leads to an increase not only in consumer surplus but in total surplus: the gain in consumer surplus ($256 - 144 = 112$) exceeds the loss in total profits ($278 - 236 = 42$).
In the above example we assumed that the two firms had the same cost function 
\( C = 10 + 2q \). However, there is no reason why this should be true. The same reasoning applies 
to the case where the **firms have different costs**. Example: demand function as before 
\( P = 50 - 2Q \) but now

\[
\begin{align*}
\text{cost function of firm 1:} & \quad C_1 = 10 + 2q_1 \\
\text{cost function of firm 2:} & \quad C_2 = 12 + 8q_2
\end{align*}
\]

Then the profit functions are:

\[
\begin{align*}
\pi_1(q_1,q_2) &= q_1 [50 - 2 (q_1 + q_2)] - 10 - 2q_1 \\
\pi_2(q_1,q_2) &= q_2 [50 - 2 (q_1 + q_2)] - 12 - 8q_2
\end{align*}
\]

The Nash equilibrium is found by solving:

\[
\begin{align*}
\frac{\partial \pi_1}{\partial q_1} &= 50 - 4q_1 - 2q_2 - 2 = 0 \\
\frac{\partial \pi_2}{\partial q_2} &= 50 - 2q_1 - 4q_2 - 8 = 0
\end{align*}
\]

The solution is \( q_1^* = 9 \), \( q_2^* = 6 \), \( Q = 15 \), \( P = 20 \), \( \pi_1 = 152 \), \( \pi_2 = 60 \). Since firms have different costs, they choose different output levels: the **low-cost firm (firm 1)** produces more 
and makes higher profits than the **high-cost firm (firm 2)**.
COURNOT OLIGOPOLY: too many firms

\[ a := 50 \quad b := 2 \quad c := 2 \quad F := 10 \]

Inverse demand
\[ P(Q) := a - b \cdot Q \quad \text{demand} \]

Cost function:
\[ C(q) := F + c \cdot q \quad \text{cost} \]

Profit function of firm 1:
\[ \Pi_1(q_1, \ldots, q_n) = q_1 \left[ 50 - 2(q_1 + \ldots + q_n) \right] - 2q_1 - 10 \]

Derivative:
\[ 50 - 2(2q_1 + q_2 + \ldots + q_n) - 2 \quad \text{Symmetric solution requires } q_1 = \ldots = q_n \]

so we have \( 48 - 2(n+1)q = 0 \)  

Thus
\[
q(n) := \frac{48}{(n+1) \cdot 2} \quad \text{firm output}
\]

\[ Q(n) := n \cdot q(n) \quad \text{industry output} \]

\[ p(n) := 50 - 2 \cdot n \cdot q(n) \quad \text{price} \]

\[ (p(n) - 2) \text{ simplify } \rightarrow \frac{48}{(n+1)} \]

\[ Pr(n) := \frac{48 \cdot 24}{(n+1)^2} - 10 \quad \text{PROFITS of each firm} \]

\[ Pr_{tot}(n) := n \cdot Pr(n) \]

\[ CS(n) := \frac{(P(0) - p(n)) \cdot Q(n)}{2} \quad \text{consumer surplus} \]

\[ CS(n) \text{ simplify } \rightarrow 576 \cdot \frac{n^2}{(n+1)^2} \]

\[ SW(n) := CS(n) + Pr_{tot}(n) \quad SW(n) \text{ simplify } \rightarrow -2 \cdot n \cdot \frac{-278 \cdot n - 571 + 5 \cdot n^2}{(n+1)^2} \quad \text{social welfare} \]
Thus (free entry) equilibrium number of firms in the industry is 9.

The socially optimum number of firms is 4.