COURNOT DUOPOLY: an example

Let the inverse demand function and the cost function be given by

\[ P = 50 - 2Q \quad \text{and} \quad C = 10 + 2q \]

respectively, where Q is total industry output and q is the firm’s output.

First consider first the case of uniform-pricing monopoly, as a benchmark. Then in this case \( Q = q \) and the profit function is

\[ \pi(Q) = (50 - 2Q)Q - 10 - 2Q = 48Q - 2Q^2 - 10. \]

Solving \( \frac{d\pi}{dQ} = 0 \) we get \( Q = 12, \ P = 26, \ \pi = 278, \ CS = \frac{12(50-26)}{2} = 144, \ TS = 278 + 144 = 422. \)

<table>
<thead>
<tr>
<th>MONOPOLY</th>
<th>Q</th>
<th>P</th>
<th>( \pi )</th>
<th>CS</th>
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<td>12</td>
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Now let us consider the case of two firms, or duopoly. Let \( q_1 \) be the output of firm 1 and \( q_2 \) the output of firm 2. Then \( Q = q_1 + q_2 \) and the profit functions are:

\[ \pi_1(q_1, q_2) = q_1 \left[ 50 - 2(q_1 + q_2) \right] - 10 - 2q_1 \]

\[ \pi_2(q_1, q_2) = q_2 \left[ 50 - 2(q_1 + q_2) \right] - 10 - 2q_2 \]

A Nash equilibrium is a pair of output levels \((q_1^*, q_2^*)\) such that:

\[ \pi_1(q_1^*, q_2^*) \geq \pi_1(q_1, q_2^*) \quad \text{for all} \ q_1 \geq 0 \]

and

\[ \pi_2(q_1^*, q_2^*) \geq \pi_2(q_1^*, q_2) \quad \text{for all} \ q_2 \geq 0. \]
This means that, fixing $q_2$ at the value $q_2^*$ and considering $\pi_1$ as a function of $q_1$ alone, this function is maximized at $q_1 = q_1^*$. But a necessary condition for this to be true is that

\[
\frac{\partial \pi_1}{\partial q_1}(q_1^*, q_2^*) = 0.
\]

Similarly, fixing $q_1$ at the value $q_1^*$ and considering $\pi_2$ as a function of $q_2$ alone, this function is maximized at $q_2 = q_2^*$. But a necessary condition for this to be true is that

\[
\frac{\partial \pi_2}{\partial q_2}(q_1^*, q_2^*) = 0.
\]

Thus the Nash equilibrium is found by solving the following system of two equations in the two unknowns $q_1$ and $q_2$:

\[
\begin{align*}
\frac{\partial \pi_1}{\partial q_1}(q_1^*, q_2^*) &= 50 - 4q_1 - 2q_2 - 2 = 0 \\
\frac{\partial \pi_2}{\partial q_2}(q_1^*, q_2^*) &= 50 - 2q_1 - 4q_2 - 2 = 0
\end{align*}
\]

The solution is $q_1^* = q_2^* = 8$, $Q = 16$, $P = 18$, $\pi_1 = \pi_2 = 118$, $CS = \frac{16(50 - 18)}{2} = 256$, $TS = 118 + 118 + 256 = 492$.

Let us compare the two.

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<tr>
<th>DUOPOLY</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>Q</th>
<th>P</th>
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<th>$\pi_2$</th>
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Thus competition leads to an increase not only in consumer surplus but in total surplus: the gain in consumer surplus ($256 - 144 = 112$) exceeds the loss in total profits ($278 - 236 = 42$).
In the above example we assumed that the two firms had the same cost function \( C = 10 + 2q \). However, there is no reason why this should be true. The same reasoning applies to the case where the **firms have different costs**. Example: demand function as before \( P = 50 - 2Q \) but now

- cost function of firm 1: \( C_1 = 10 + 2q_1 \)
- cost function of firm 2: \( C_2 = 12 + 8q_2 \).

Then the profit functions are:

\[
\pi_1(q_1, q_2) = q_1 \left[ 50 - 2(q_1 + q_2) \right] - 10 - 2q_1
\]

\[
\pi_2(q_1, q_2) = q_2 \left[ 50 - 2(q_1 + q_2) \right] - 12 - 8q_2
\]

The Nash equilibrium is found by solving:

\[
\begin{align*}
\frac{\partial \pi_1}{\partial q_1}(q_1^*, q_2^*) &= 50 - 4q_1 - 2q_2 - 2 = 0 \\
\frac{\partial \pi_2}{\partial q_2}(q_1^*, q_2^*) &= 50 - 2q_1 - 4q_2 - 8 = 0
\end{align*}
\]

The solution is \( q_1^* = 9, q_2^* = 6 \), \( Q = 15, P = 20, \pi_1 = 152, \pi_2 = 60 \). Since firms have different costs, they choose different output levels: the **low-cost firm (firm 1)** produces more and makes higher profits **than the high-cost firm (firm 2)**.