ANSWERS TO PRACTICE PROBLEMS 19

1. The core is the set of (x_1, x_2, x_3) such that

$x_1 \geq v(\{1\}) = 10$	(1)
$x_2 \geq v(\{2\}) = 6$	(2)
$x_3 \geq v(\{3\}) = 8$	(3)
$x_1 + x_2 \geq v(\{1,2\}) = 18$	(4)
$x_1 + x_3 \geq v(\{1,3\}) = 24$	(5)
$x_2 + x_3 \geq v(\{2,3\}) = 16$	(6)
$x_1 + x_2 + x_3 = v(\{1,2,3\}) = 30$	(7)

From (5) and (7) we get that $x_2 \leq 6$. This, together with (2), gives

$$x_2 = 6. \quad (8)$$

From (7) and (8) we get that $x_1 + x_3 = 24$ so that

$$x_3 = 24 - x_1. \quad (9)$$

From (4) and (8) we get that

$$x_1 \geq 12. \quad (10).$$

From (6) and (8) we get that $x_3 \geq 10$ and this, together with (9) gives $x_1 \leq 14$.

Thus the core is the set of triples (x_1, x_2, x_3) such that $12 \leq x_1 \leq 14$, $x_2 = 6$ and $x_3 = 24 - x_1$.

2. The core is the set of (x_1, x_2, x_3) such that

$x_1 \geq v(\{1\}) = 0$	(1)
$x_2 \geq v(\{2\}) = 0$	(2)
$x_3 \geq v(\{3\}) = 0$	(3)
$x_1 + x_2 \geq v(\{1,2\}) = 40$	(4)
$x_1 + x_3 \geq v(\{1,3\}) = 0$	(5)
$x_2 + x_3 \geq v(\{2,3\}) = 50$	(6)
$x_1 + x_2 + x_3 = v(\{1,2,3\}) = 50$	(7)

From (6) and (7) we get that $x_1 \leq 0$. This, together with (1), gives

$$x_1 = 0. \quad (8)$$

From (7) and (8) we get that $x_2 + x_3 = 50$ so that

$$x_3 = 50 - x_2. \quad (9)$$

From (4) and (8) we get that

$$x_2 \geq 40. \quad (10)$$

Thus the core is the set of triples (x_1, x_2, x_3) such that $x_1 = 0$, $x_2 \geq 40$ and $x_3 = 50 - x_2$.

3. **(a)** The core is the set of (x_1, x_2) such that $x_1 \geq 2$, $x_2 \geq 5$ and $x_1 + x_2 = 8$. Thus the set of pairs $(x_1, 8 - x_1)$ such that $2 \leq x_1 \leq 3$.

(b) Only two: (2, 6) and (3,5).

4.

1. (6, 6, 6) is not in the core because, for example, the coalition {1,2} can block it with (7,7,0).
2. (4, 6, 8) is not in the core because, for example, the coalition {1,2} can block it with (7,7,0).
3. (7, 7, 4) is not in the core because, for example, the coalition {2,3} can block it with (0,8,8).
4. (8, 8, 2) is not in the core because, for example, the coalition {2,3} can block it with (0,9,7).

5. If (x_1, x_2, x_3) is in the core it must satisfy the following inequalities:

(1) $x_1 + x_2 \geq 12$, (2) $x_1 + x_3 \geq 10$, (3) $x_2 + x_3 \geq 14$

Adding these inequalities we get $2x_1 + 2x_2 + 2x_3 \geq 36$, that is, $x_1 + x_2 + x_3 \geq 18$ which is impossible since v({1,2,3}) = 16.

6.

1. (4, 4, 5, 5) is not in the core because, for example, the coalition {3,4} can block it with (0,0,6,6).
2. (2, 4, 6, 6) is not in the core because, for example, the coalition {1,2} can block it with (3,5,0,0).
3. (4, 5, 5, 4) is not in the core because, for example, the coalition {3,4} can block it with (0,0,6,6).

7. No, the core is empty because, in order to be in the core, an imputation (x_1, x_2, x_3, x_4) must be such that $x_3 + x_4 \geq 12$ (otherwise it can be blocked by the coalition {3,4}) and, furthermore, it must be such that $x_1 \geq 4$ (otherwise it can be blocked by the coalition {1}) and $x_2 \geq 4$ (otherwise it can be blocked by the coalition {2}), so that $x_1 + x_2 + x_3 + x_4 \geq 20$, which is impossible, since v(N) = 18.

8. The Shapley value is $x_1 = 14$, $x_2 = 11$, $x_2 = 9$ and is calculated as follows:

v({1})	v({2})	v({3})	v({1,2})	v({1,3})	v({2,3})	v({1,2,3})
10	8	6	24	22	18	34
order	**probability**	**player 1's marginal contribution**	**player 2's marginal contribution**	**player 3's marginal contribution**		
123	1/6	10	14	10		
132	1/6	10	12	12		
213	1/6	16	8	10		
231	1/6	16	8	10		
312	1/6	16	12	6		
321	1/6	16	12	6		
	sum	**84**	**66**	**54**		
					check sum	
	Shapley value	**14**	**11**	**9**	34	

9. The Shapley value is $x_1 = 115$, $x_2 = 85$, $x_2 = 60$ and is calculated as follows:

v({1})	v({2})	v({3})	v({1,2})	v({1,3})	v({2,3})	v({1,2,3})
80	60	30	180	160	120	260
order	**probability**	**player 1's marginal contribution**	**player 2's marginal contribution**	**player 3's marginal contribution**		
123	1/6	80	100	80		
132	1/6	80	100	80		
213	1/6	120	60	80		
231	1/6	140	60	60		
312	1/6	130	100	30		
321	1/6	140	90	30		
	sum	**690**	**510**	**360**		
					check sum	
	Shapley value	**115**	**85**	**60**	260	

10. Player 1 is not a dummy player, because v({1,2}) – v({2}) = 180 – 60 = 120 > v({1}) = 80.

11. Players 1 and 2 are not interchangeable because v({1}) ≠ v({2}).

12. **(a)** Players 1 and 3 are interchangeable because v({1}) = v({3}) and v({1,2}) – v({2}) = v({2,3}) – v({3}) = 4.

(b) The Shapley value is (4, 4, 4).

(c) The Shapley value is not in the core because (4, 4, 4) can be blocked by the coalition {1,3} with (5, 0, 5)

13. **(a)** No two players are interchangeable because v({i}) ≠ v({j}) for any i ≠ j.

(b) Player 1 is a dummy player because v({1,2}) = v({2}) + v({1}), v({1,3}) = v({3}) + v({1}) and v({1,2,3}) = v({2,3}) + v({1}).

(c) The Shapley value is (2, 5, 7).

(d) The Shapley value is in the core because it satisfies all the inequalities that define the core.