Department of Economics, University of California, Davis
Ecn 200C – Micro Theory – Professor Giacomo Bonanno

ANSWERS TO PRACTICE PROBLEMS $\boxed{8}$

1. **(a)** Let firm 1 be located at $\frac{1}{4}$ and firm 2 at 1. Let p_1 be the price of firm 1 and p_2 the price of firm 2. Let x be the location of the consumer who is indifferent between buying from firm 1 and buying from firm 2. Then x must satisfy:

$$p_1 + 2\left(x - \frac{1}{4}\right) = p_2 + 2(1 - x).$$

Hence $x = \frac{p_2 - p_1}{4} + \frac{5}{8}$. Demand for firm 1 is x, while demand for firm 2 is $(1-x)$. Thus:

$$D_1(p_1,p_2) = \frac{p_2 - p_1}{4} + \frac{5}{8}$$

$$D_2(p_1,p_2) = \frac{3}{8} + \frac{p_1 - p_2}{4}.$$

(b) The profit (= revenue) function of firm 1 is given by $\pi_1 = p_1 D_1$ and the profit (= revenue) function of firm 2 is given by $\pi_2 = p_2 D_2$. The Nash equilibrium is obtained by solving the system of equations $\frac{\partial \pi_1}{\partial p_1} = 0$ and $\frac{\partial \pi_2}{\partial p_2} = 0$.

$$\frac{\partial \pi_1}{\partial p_1} = \frac{p_2 - p_1}{4} + \frac{5}{8} - \frac{p_1}{4} = 0$$

$$\frac{\partial \pi_2}{\partial p_2} = \frac{3}{8} + \frac{p_1 - p_2}{4} - \frac{p_2}{4} = 0.$$

The solution is $p_1 = \frac{13}{6}$ and $p_2 = \frac{11}{6}$. Thus the two gas stations would charge different prices.

2. Let p be the common price. Let firm 1 be located at x_1 and firm 2 at x_2 ($0 \leq x_1 \leq x_2 \leq 1$).

Then the indifferent consumer is located half-way between the two firms at $\frac{x_1 + x_2}{2}$. Hence the profit functions are given by:

$$\pi_1(x_1,x_2) = \frac{x_1 + x_2}{2}\, p$$

$$\pi_2(x_1,x_2) = \left[1 - \frac{x_1 + x_2}{2}\right] \text{p.}$$

Since $\frac{\partial \pi_1}{\partial x_1} = \frac{p}{2} > 0$ and $\frac{\partial \pi_2}{\partial x_2} = -\frac{p}{2} < 0$, firm 1 can increase its profits by moving towards firm 2 and firm 2 can increase its profits by moving toward firm 1. Hence the only candidates for Nash equilibria are the pairs (x_1,x_2) with $x_1 = x_2 = x$ (so that each firm gets half of the market and makes a profit of $\frac{p}{2}$. If $x < \frac{1}{2}$, firm 1 can increase its profits by jumping slightly to the right of firm 2 (thereby capturing more that half of the market). Similarly, if $x > \frac{1}{2}$, firm 2 can increase its profits by jumping slightly to the left of firm 1. Hence there is a unique Nash equilibrium where both firms locate at $x = \frac{1}{2}$.

3. There is no Nash equilibrium. *Proof.* Suppose there is a Nash equilibrium (p_1^*, p_2^*, x_1^*, x_2^*). We shall consider all the possible cases.

Case (1): $p_1^* = p_2^* > 0$. Then we know that (see problem 1 above) unless $x_1^* = x_2^* = \frac{1}{2}$, one firm can increase its profits by changing its location. However, $p_1^* = p_2^*$ and $x_1^* = x_2^* = \frac{1}{2}$ is not a Nash equilibrium because one firm can undercut its rival by a tiny amount and get the whole market.

Case (2): $p_1^* > p_2^* > 0$. If $x_1^* = x_2^*$ then firm 1 is making zero profits and can make a profit of $p_2^*/2$ by reducing its price to p_2^*. If $x_1^* \neq x_2^*$ and firm 1's demand is $D_1^* = 0$ the same argument applies. If, on the other hand, firm 1's demand is $D_1^* > 0$ then firm 2's profits are $\pi_2^* = (1 - D_1^*)\, p_2^*$ and firm 2 can increase its profits by setting $x_2 = x_1^*$.

Case (3): $p_2^* > p_1^* > 0$. Similar reasoning.

Remaining cases: if $p_i^* = 0$ for some $i = 1,2$, then firm i can increase its profits by increasing its price and, if necessary, moving away from the other firm.

4. The Nash equilibria are given by all pairs (p_1,p_2) such that $4 \le p_1 = p_2 \le 6$.

Proof: First of all we show that any (p_1,p_2) with $4 \le p_1 = p_2 \le 6$ is a Nash equilibrium. Firm 2 makes zero profits. If it reduces its price it will make a loss, if it increases its price it will still make zero profits. Firm 1's profits are $2\,(p_1 - 4) \ge 0$. If it reduces its price, profits will go down, if it increases its price profits will be zero.

Now we show that $p_1 = p_2 > 6$ is not a Nash equilibrium. Firm 2 is making zero profits and it can increase its profits by charging a price p_2^* such that $6 < p_2^* < p_1$.

(p_1,p_2) with $p_1 < 4$ and $p_1 \le p_2$ is not a Nash equilibrium because firm 1 makes a loss and can increase its profits to zero by increasing p_1. Finally:

(A) $4 \le p_1 < 10$ and $p_1 < p_2$ is not a Nash equilibrium because firm 1 can increase its profits by increasing p_1;

(B) $p_1 = 10 < p_2$ is not a Nash equilibrium because firm 2 can make positive profits by reducing p_2 below p_1;

(C) $10 < p_1 < p_2$ is not a Nash equilibrium because firm 1 can increase its profits by reducing p_1;

(D) $p_2 < 6$ and $p_2 < p_1$ is not a Nash equilibrium because firm 2 is making a loss;

(E) $p_2 \geq 6$ and $p_2 < p_1$ is not a Nash equilibrium because firm 1 can increase its profits by reducing its price to p_2.

5. First we calculate the Bertrand-Nash equilibrium. Firm i's profit function is given by $\pi_i = p_i q_i$. Solving the system of equations $\frac{\partial \pi_i}{\partial p_i} = 0$ (i=1,2) we obtain:

$$p_1^B = \frac{2(a-1)(a-b)}{a(4a-b-3)}, \quad q_1^B = \frac{2(a-1)}{(4a-b-3)}, \quad \pi_1^B = \frac{4(a-1)^2(a-b)}{a(4a-b-3)^2},$$

$$p_2^B = \frac{(b-1)(a-b)}{b(4a-b-3)}, \quad q_2^B = \frac{(a-1)}{(4a-b-3)}, \quad \pi_2^B = \frac{(a-1)(a-b)(b-1)}{b(4a-b-3)^2}$$

Similarly, the Cournot-Nash equilibrium is given by:

$$p_1^C = \frac{(a-1)(2a-b-1)}{a(4a-b-3)}, \quad q_1^C = \frac{(2a-b-1)}{(4a-b-3)}, \quad \pi_1^C = \frac{(a-1)(2a-b-1)^2}{a(4a-b-3)^2}$$

$$p_2^C = \frac{(b-1)(a-1)}{b(4a-b-3)}, \quad q_2^C = \frac{(a-1)}{(4a-b-3)}, \quad \pi_2^C = \frac{(a-1)^2(b-1)}{b(4a-b-3)^2},$$

It is easy to check that for i = 1,2 $p_i^B < p_i^C$; $q_i^B \geq q_i^C$ (more precisely, $q_1^B > q_1^C$ and $q_2^B = q_2^C$); $\pi_i^B < \pi_i^C$.

Thus prices and profits are lower at the Bertrand-Nash equilibrium than at the Cournot-Nash equilibrium.