1. (a) Let firm 1 be located at \( \frac{1}{4} \) and firm 2 at 1. Let \( p_1 \) be the price of firm 1 and \( p_2 \) the price of firm 2. Let \( x \) be the location of the consumer who is indifferent between buying from firm 1 and buying from firm 2. Then \( x \) must satisfy:

\[
p_1 + 2 \left( x - \frac{1}{4} \right) = p_2 + 2 (1 - x).
\]

Hence \( x = \frac{p_2 - p_1}{4} + \frac{5}{8} \). Demand for firm 1 is \( x \), while demand for firm 2 is \( (1-x) \).

Thus:

\[
D_1(p_1,p_2) = \frac{p_2 - p_1}{4} + \frac{5}{8}
\]

\[
D_2(p_1,p_2) = \frac{3}{8} + \frac{p_1 - p_2}{4}.
\]

(b) The profit (= revenue) function of firm 1 is given by \( \pi_1 = p_1 D_1 \) and the profit (= revenue) function of firm 2 is given by \( \pi_2 = p_2 D_2 \). The Nash equilibrium is obtained by solving the system of equations \( \frac{\partial \pi_1}{\partial p_1} = 0 \) and \( \frac{\partial \pi_2}{\partial p_2} = 0 \).

\[
\frac{\partial \pi_1}{\partial p_1} = \frac{p_2 - p_1}{4} + \frac{5}{8} - \frac{p_1}{4} = 0
\]

\[
\frac{\partial \pi_2}{\partial p_2} = \frac{3}{8} + \frac{p_1 - p_2}{4} - \frac{p_2}{4} = 0.
\]

The solution is \( p_1 = \frac{13}{6} \) and \( p_2 = \frac{11}{6} \). Thus the two gas stations would charge different prices.
Let $p$ be the common price. Let firm 1 be located at $x_1$ and firm 2 at $x_2$ ($0 \leq x_1 \leq x_2 \leq 1$). Then the indifferent consumer is located half-way between the two firms at $\frac{x_1 + x_2}{2}$. Hence the profit functions are given by:

\[
\pi_1(x_1, x_2) = \frac{x_1 + x_2}{2} p
\]

\[
\pi_2(x_1, x_2) = [1 - \frac{x_1 + x_2}{2}] p.
\]

Since $\frac{\partial \pi_1}{\partial x_1} = \frac{p}{2} > 0$ and $\frac{\partial \pi_2}{\partial x_2} = -\frac{p}{2} < 0$, firm 1 can increase its profits by moving towards firm 2 and firm 2 can increase its profits by moving toward firm 1. Hence the only candidates for Nash equilibria are the pairs $(x_1, x_2)$ with $x_1 = x_2 = x$ (so that each firm gets half of the market and makes a profit of $\frac{p}{2}$. If $x < \frac{1}{2}$, firm 1 can increase its profits by jumping slightly to the right of firm 2 (thereby capturing more than half of the market). Similarly, if $x > \frac{1}{2}$, firm 2 can increase its profits by jumping slightly to the left of firm 1. Hence there is a unique Nash equilibrium where both firms locate at $x = \frac{1}{2}$.

3. There is no Nash equilibrium. Proof. Suppose there is a Nash equilibrium $(p_1^*, p_2^*, x_1^*, x_2^*)$. We shall consider all the possible cases.

Case (1): $p_1^* = p_2^* > 0$. Then we know that (see problem 1 above) unless $x_1^* = x_2^* = \frac{1}{2}$, one firm can increase its profits by changing its location. However, $p_1^* = p_2^*$ and $x_1^* = x_2^* = \frac{1}{2}$ is not a Nash equilibrium because one firm can undercut its rival by a tiny amount and get the whole market.
Case (2): $p_1^* > p_2^* > 0$. If $x_1^* = x_2^*$ then firm 1 is making zero profits and can make a profit of $p_2^*/2$ by reducing its price to $p_2^*$. If $x_1^* \neq x_2^*$ and firm 1's demand is $D_1^* = 0$ the same argument applies. If, on the other hand, firm 1's demand is $D_1^* > 0$ then firm 2's profits are $\pi_2^* = (1 - D_1^*) p_2^*$ and firm 2 can increase its profits by setting $x_2 = x_1^*$.

Case (3): $p_2^* > p_1^* > 0$. Similar reasoning.

Remaining cases: if $p_i^* = 0$ for some $i = 1,2$, then firm $i$ can increase its profits by increasing its price and, if necessary, moving away from the other firm.

4. The Nash equilibria are given by all pairs $(p_1,p_2)$ such that $4 \leq p_1 = p_2 \leq 6$.

Proof: First of all we show that any $(p_1,p_2)$ with $4 \leq p_1 = p_2 \leq 6$ is a Nash equilibrium. Firm 2 makes zero profits. If it reduces its price it will make a loss, if it increases its price it will still make zero profits. Firm 1’s profits are $2 (p_1 - 4) \geq 0$. If it reduces its price, profits will go down, if it increases its price profits will be zero.

Now we show that $p_1 = p_2 > 6$ is not a Nash equilibrium. Firm 2 is making zero profits and it can increase its profits by charging a price $p_2^*$ such that $6 < p_2^* < p_1$.

$(p_1,p_2)$ with $p_1 < 4$ and $p_1 \leq p_2$ is not a Nash equilibrium because firm 1 makes a loss and can increase its profits to zero by increasing $p_1$. Finally:

(A) $4 \leq p_1 < 10$ and $p_1 < p_2$ is not a Nash equilibrium because firm 1 can increase its profits by increasing $p_1$;

(B) $p_1 = 10 < p_2$ is not a Nash equilibrium because firm 2 can make positive profits by reducing $p_2$ below $p_1$;

(C) $10 < p_1 < p_2$ is not a Nash equilibrium because firm 1 can increase its profits by reducing $p_1$;

(D) $p_2 < 6$ and $p_2 < p_1$ is not a Nash equilibrium because firm 2 is making a loss;
(E) \( \ p_2 \geq 6 \) and \( \ p_2 \ < \ p_1 \) is not a Nash equilibrium because firm 1 can increase its profits by reducing its price to \( \ p_2 \).

5. First we calculate the Bertrand-Nash equilibrium. Firm i’s profit function is given by

\[
\pi_i = p_i q_i.
\]

Solving the system of equations \( \frac{\partial \pi_i}{\partial p_i} = 0 \) (i=1,2) we obtain:

\[
\begin{align*}
p_1^B &= \frac{2(a-1)(a-b)}{a(4a-b-3)} , \\
q_1^B &= \frac{2(a-1)}{(4a-b-3)} , \\
\pi_1^B &= \frac{4(a-1)^2(a-b)}{a(4a-b-3)^2} , \\
p_2^B &= \frac{(b-1)(a-b)}{b(4a-b-3)} , \\
q_2^B &= \frac{(a-1)}{(4a-b-3)} , \\
\pi_2^B &= \frac{(a-1)(a-b)(b-1)}{b(4a-b-3)^2}
\end{align*}
\]

Similarly, the Cournot-Nash equilibrium is given by:

\[
\begin{align*}
p_1^C &= \frac{(a-1)(2a-b-1)}{a(4a-b-3)} , \\
q_1^C &= \frac{(2a-b-1)}{(4a-b-3)} , \\
\pi_1^C &= \frac{(a-1)(2a-b-1)^2}{a(4a-b-3)^2} , \\
p_2^C &= \frac{(b-1)(a-b)}{b(4a-b-3)} , \\
q_2^C &= \frac{(a-1)}{(4a-b-3)} , \\
\pi_2^C &= \frac{(a-1)^2(b-1)}{b(4a-b-3)^2}
\end{align*}
\]

It is easy to check that for \( i = 1,2 \) \( p_i^B \ < \ p_i^C ; \ q_i^B \ \geq \ q_i^C \) (more precisely, \( q_1^B \ > \ q_1^C \) and \( q_2^B \ = \ q_2^C \); \( \pi_i^B \ < \ \pi_i^C \).

Thus prices and profits are lower at the Bertrand-Nash equilibrium than at the Cournot-Nash equilibrium.