Practice problems on cooperative games

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200C – Micro Theory – Professor Giacomo Bonanno

Topic: Cooperative games
(core and Shapley value)

VERY IMPORTANT: do not look at the answers until you have made a VERY serious effort to solve the problem. If you turn to the answers to get clues or help, you are wasting a chance to test how well you are prepared for the exams. I will not give you more practice problems later on.

1. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
   \[
   \begin{align*}
   v(\{1\}) &= 10, \quad v(\{2\}) = 6, \quad v(\{3\}) = 8 \\
   v(\{1,2\}) &= 18, \quad v(\{1,3\}) = 24, \quad v(2,3) = 16 \\
   v(\{1,2,3\}) &= 30.
   \end{align*}
   
   Find the core.

2. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
   \[
   \begin{align*}
   v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \\
   v(\{1,2\}) &= 40, \quad v(\{1,3\}) = 0, \quad v(2,3) = 50 \\
   v(\{1,2,3\}) &= 50
   \end{align*}
   
   Find the core.

3. Consider the following cooperative game: \( N = \{1, 2\} \) and
   \[
   v(\{1\}) = 2, \quad v(\{2\}) = 5, \quad v(\{1,2\}) = 8.
   
   (a) Find the core.

   (b) If imputations are required to be (component-by-component) integer-valued, list all the imputations in the core.
4. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
\[
\begin{align*}
  v(\{1\}) &= 4, \quad v(\{2\}) = 6, \quad v(\{3\}) = 3 \\
  v(\{1,2\}) &= 14, \quad v(\{1,3\}) = 12, \quad v(\{2,3\}) = 16 \\
  v(\{1,2,3\}) &= 18
\end{align*}
\]
For each of the following imputations \((x_1, x_2, x_3)\) determine if it is in the core:
1. \((6, 6, 6)\)
2. \((4, 6, 8)\)
3. \((7, 7, 4)\)
4. \((8, 8, 2)\)

5. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
\[
\begin{align*}
  v(\{1\}) &= 2, \quad v(\{2\}) = 4, \quad v(\{3\}) = 1 \\
  v(\{1,2\}) &= 12, \quad v(\{1,3\}) = 10, \quad v(\{2,3\}) = 14 \\
  v(\{1,2,3\}) &= 16
\end{align*}
\]
Prove that the core is empty.

6. Consider the following cooperative game: \( N = \{1, 2, 3, 4\} \) and
\[
\begin{align*}
  v(\{1\}) &= v(\{2\}) = 4, \quad v(\{3\}) = v(\{4\}) = 6 \\
  v(\{1,2\}) &= v(\{1,3\}) = v(\{1,4\}) = 8, \quad v(\{2,3\}) = 10, \quad v(\{2,4\}) = 10, \quad v(\{3,4\}) = 12, \\
  v(\{1,2,3\}) &= v(\{1,2,4\}) = v(\{2,3,4\}) = 14, \\
  v(\{1,2,3,4\}) &= 18
\end{align*}
\]
For each of the following imputations \((x_1, x_2, x_3, x_4)\) determine if it is in the core:
1. \((4, 4, 5, 5)\)
2. \((2, 4, 6, 6)\)
3. \((4, 5, 5, 4)\)
7. Consider the following cooperative game: \( N = \{1, 2, 3, 4\} \) and
\[
\begin{align*}
v(\{1\}) &= v(\{2\}) = 4, \quad v(\{3\}) = v(\{4\}) = 6 \\
v(\{1,2\}) &= v(\{1,3\}) = v(\{1,4\}) = 8, \quad v(\{2,3\}) = 10, \quad v(\{2,4\}) = 10, \quad v(\{3,4\}) = 12, \\
v(\{1,2,3\}) &= v(\{1,2,4\}) = v(\{2,3,4\}) = 14, \\
v(\{1,2,3,4\}) &= 18
\end{align*}
\]
Is the core non-empty?

8. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
\[
\begin{align*}
v(\{1\}) &= 10, \quad v(\{2\}) = 8, \quad v(\{3\}) = 6 \\
v(\{1,2\}) &= 24, \quad v(\{1,3\}) = 22, \quad v(\{2,3\}) = 18 \\
v(\{1,2,3\}) &= 34
\end{align*}
\]
Find the Shapley value.

9. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and
\[
\begin{align*}
v(\{1\}) &= 80, \quad v(\{2\}) = 60, \quad v(\{3\}) = 30 \\
v(\{1,2\}) &= 180, \quad v(\{1,3\}) = 160, \quad v(\{2,3\}) = 120 \\
v(\{1,2,3\}) &= 260
\end{align*}
\]
Find the Shapley value.

10. Consider again the game of Exercise 9. Is Player 1 a dummy player?
11. Consider again the game of Exercise 9. Are Players 1 and 2 interchangeable?

12. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and

\[
\begin{align*}
  v(\{1\}) &= 2, & v(\{2\}) &= 4, & v(\{3\}) &= 2 \\
  v(\{1,2\}) &= 8, & v(\{1,3\}) &= 10, & v(\{2,3\}) &= 8 \\
  v(\{1,2,3\}) &= 12
\end{align*}
\]

(a) Are Players 1 and 3 interchangeable?

(b) Find the Shapley value.

(c) Is the Shapley value in the core?

13. Consider the following cooperative game: \( N = \{1, 2, 3\} \) and

\[
\begin{align*}
  v(\{1\}) &= 2, & v(\{2\}) &= 4, & v(\{3\}) &= 6 \\
  v(\{1,2\}) &= 6, & v(\{1,3\}) &= 8, & v(\{2,3\}) &= 12 \\
  v(\{1,2,3\}) &= 14
\end{align*}
\]

(a) Are any two players interchangeable?

(b) Is any player a dummy player?

(c) Find the Shapley value.

(d) Is the Shapley value in the core?