Sequential Equilibrium

The notion of sequential equilibrium is due to Kreps and Wilson (1982). To motivate it, we shall look at a number of games.

Consider first the following game.

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NATURE
A 1/3 B 2/3
C       D
X1 x1 1 X2 x2
D       D
X3 x3 X4 x4
E F E F
Z2 z2 Z3 z3 Z4 z4 Z5 z5
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Suppose that the payoffs are such that there is a Nash equilibrium (perhaps unique) at which player 1 plays C [ask a student to give such payoffs]. Thus 2’s information set is not reached and the notion of weak sequential equilibrium does not impose any restrictions on player 2’s beliefs at her information set. It seems clear, however, that the only reasonable beliefs are: \( x_3 \) with probability \( 1/3 \) and \( x_4 \) with probability \( 2/3 \). These are indeed the probabilities required by the notion of sequential equilibrium.

Consider now the following game.

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The following is a weak sequential equilibrium: \( \sigma = ((R,B,D), \mu[x_1] = 1, \mu[x_2] = 0, \mu[x_3] = 0, \mu[x_4] = 1 \). Player 2’s information set \( h \) is not reached in equilibrium, hence the notion
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of weak sequential equilibrium imposes no restrictions on the beliefs there. In particular \( \text{Prob}(x_1) = 1 \) and \( \text{Prob}(x_2) = 0 \) is acceptable. However, the beliefs of player 2 at \( g \) seem to be inconsistent with her previous beliefs at \( h \). At \( g \) player 2 believes that node \( x_4 \) was reached. This means that she believes that player 1 chose \( M \). On the other hand, at \( h \) she believed that \( x_1 \) was reached, that is, that player 1 chose \( L \). Now the only basis for changing her mind about what player 1 did is the fact that she herself switched from the equilibrium \( B \) to \( A \). Many people would consider this switching of beliefs of player 2 irrational or unjustified. While the notion of weak sequential equilibrium allows it, the notion of sequential equilibrium does not.

Consider now the following game.

![Game Diagram]

The following is a weak sequential equilibrium \((C,D,G,M)\) with beliefs: \( \mu[x_2] = \mu[x_3] = \mu[x_4] = 1 \). Since the information sets of players 2, 3 and 4 are not reached in equilibrium (and the equilibrium choice of player 2 is \( D \)), the notion of weak sequential equilibrium does not impose any restrictions on beliefs. However, the beliefs of player 3 and 4 reveal a disagreement between the two players: player 3 believes that 2 chose \( E \), while player 4 believes that 2 chose \( F \). The notion of sequential equilibrium does not allow such disagreement.

The notion of sequential equilibrium imposes strong restrictions on beliefs at information sets out of the equilibrium path. Thus the notion of sequential equilibrium adds to the notion of weak sequential equilibrium strong requirements concerning beliefs at information sets which are reached with zero probability. As before, an assessment is defined as a pair \((\mu, \sigma)\), where \( \sigma \) is an \( n \)-tuple of behavior strategies and \( \mu \) is a system of beliefs. As before we require that \( \mu \) satisfy Bayes' rule at every information set which is reached with positive probability by \( \sigma \). For information sets which are not reached when \( \sigma \) is played, we proceed as follows.

Let \( \Sigma_0 \) be the set of \( n \)-tuples of behavior strategies which assign positive probability to \textit{every} actions at \textit{every} information set. Then, given a \( \sigma \in \Sigma_0 \), Bayes' rule yields a unique
probability $\mu(x)$ for each decision node $x$. Let $A_0$ be the set of assessments $(\mu,\sigma)$ such that $\sigma \in \Sigma_0$ and $\mu$ is obtained from $\sigma$ via Bayes' rule.

**DEFINITION**: an assessment $(\mu,\sigma)$ is **consistent** if $(\mu,\sigma) = \lim_{n \to \infty} (\mu_n, \sigma_n)$ for some sequence $\langle (\mu_n, \sigma_n) \rangle \in A_0$.

**DEFINITION** (as before): An assessment $(\mu,\sigma)$ is **sequentially rational** if for each player $i$, taking as fixed the system of beliefs $\mu$ and the strategies of the other players $\sigma_{-i}$, $\sigma_i$ maximizes player $i$'s payoff.

**DEFINITION**: A **sequential equilibrium** is an assessment $(\mu,\sigma)$ that is both consistent and sequentially rational.

**THEOREM** (Kreps and Wilson, 1982). Every finite, extensive-form game with perfect recall has at least one sequential equilibrium. Furthermore, if $(\mu,\sigma)$ is a sequential equilibrium then it is a weak sequential equilibrium.
Example 1

Before we give an example of a sequential equilibrium, let us examine the intuitive content of this equilibrium notion. It can be put as follows:

1. players start with a given Nash equilibrium as their original hypothesis as to how the game will be played;
2. if they observe events which are consistent with their original hypothesis they stick to it;
3. if they observe an event which had zero probability given the original hypothesis, then they will explain the deviation as a mistake on the part of one (or more) of the players. They will then formulate an assessment of the present situation (a subjectively probabilistic hypothesis of the mistake which was made) and choose an action which maximizes payoff on the assumption that further deviations are impossible. Note: further mistakes are assumed to be impossible as opposed to just very unlikely.

The following example illustrates the above, in particular point 3.

This game has 8 Nash equilibria in pure strategies: (AL,al),  (AL,ar),  (AL,dl),  (AL,dr),  (AR,al),  (AR,ar),  (AR,dl),  (AR,dr).

Let us show that  (AR,dl) with beliefs $\mu(x) = 0$, $\mu(y) = 1$ is a sequential equilibrium. If it is, then at the singleton information sets player 2 is saying to himself:

"Player 1 should have played A according to the original hypothesis, but in fact he played D. Hence he made a mistake. However, from this observation I don't draw any inference about his rationality, in particular I expect him to stick to the remaining part of the original hypothesis, namely R. Not only this, but I don't expect him to make another mistake, not even with arbitrarily small probability $\delta > 0$. If I did expect him to make another mistake, I should choose a which gives me 2 for sure rather than dl which gives me at most 2−2$\delta$."

Let $s^* = A \ D \ L \ R \ a \ d \ l \ r$

Let $s^* = 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0$
Let $\mu^*$ be given as follows: $\mu^*(x) = 0$, $\mu^*(y) = 1$ and for every other node $t$, $\mu^*(t) = 1$

Let $\sigma_n = \begin{array}{cccc}
A & D & L & R \\
1 - \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & 1 - \frac{1}{n} \\
\end{array}$

Clearly, $\sigma_n \rightarrow s^*$.

Now, $P(x \text{ or } y) = P(D) P(d) \left[ P(L) + P(R) \right] = \frac{1}{n} \left( 1 - \frac{1}{n} \right)$

$\mu_n(x) = P(x \mid x \text{ or } y) = \frac{\frac{1}{n} \left( 1 - \frac{1}{n} \right)}{\frac{1}{n} \left( 1 - \frac{1}{n} \right)} = \frac{1}{n}$

$\mu_n(y) = P(y \mid x \text{ or } y) = 1 - 1/n$

Then $\mu_n \rightarrow \mu^*$. Thus $(s^*,\mu^*)$ satisfies consistency. It is also sequentially rational. In fact, $(AR)$ is clearly a payoff-maximizing strategy; at the second information set of player 1 $E(R) = 0$, $E(L) = -1$; at the singleton information set of player 2 $E(al)=E(ar)=E(dl)=2$ $E$(anything else)$<2$, etc.

**Example 2**

In this game $(T,L,b ; l)$ is a Nash equilibrium. However, there is no sequential equilibrium where player 1 chooses $T$ with probability 1. Recall that a sequential equilibrium is defined in terms of a pair $(\sigma,\mu)$ where $\sigma=(\sigma_1,\sigma_2)$ is a (behavior) strategy profile and $\mu : \{x_0,x_1,x_2,x_3,x_4,x_5\} \rightarrow [0,1]$ is a function satisfying $\mu(x_0)=\mu(x_1)=\mu(x_2)=1$ and $\mu(x_3)+\mu(x_4)+\mu(x_5)=1$. We shall write $\sigma_i(a)=p$ to mean that player $i$'s strategy $\sigma_i$ assigns probability $p$ to choice $a$. Now, suppose there is a sequential equilibrium $(\sigma,\mu)$ with $\sigma_1(T)=1$. Then it must be $\sigma_2(l)>\frac{1}{2}$ (otherwise $B$ would be player 1's unique best choice), which implies that $\sigma_1(b)=1$ and $\sigma_1(L)=1$. Thus $\sigma$ must be of the form $((T,L,b),(p,1-p))$, where $p>\frac{1}{2}$ is the probability with which player 2 chooses $l$. Now consider a sequence $\{\sigma^1,\sigma^2,\ldots\}$ of completely mixed strategies whose $m^{th}$ element is given by

$$\sigma^m=(1-a_m-b_m; a_m; b_m; 1-c_m; c_m; 1-d_m; d_m; (1-e_m; e_m))$$

$T$ $M$ $B$ $L$ $R$ $b$ $t$ $l$ $r$

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with \( a_m, b_m, c_m, d_m \in (0, 1) \) and converging to zero as \( m \) tends to infinity, while \( e_m \in (0, 1) \) converges to \( 1 - p \). Let \( h \) be the information set of player 2. Now, \( \text{Prob}\{h \mid \sigma^m\} = b_m + a_m(1 - c_m d_m) \) and \( \text{Prob}\{x_5 \mid \sigma^m\} = a_m c_m(1 - d_m) \). Thus, given \( \sigma^m \), \( \text{Prob}\{x_5 \mid h\} = \left( \frac{c_m(1 - d_m)}{b_m + 1 - c_m d_m} \right) \).

As \( m \) goes to infinity, the numerator tends to zero while the denominator either tends to infinity or converges to a number greater than or equal to 1. Hence \( \mu(x_5) = 0 \), that is, player 2 must attach zero probability to node \( x_5 \) if his information set is reached. But \( \mu(x_5) = 0 \) requires player 2 to play \( l \) with probability zero, a contradiction.

Intuitively, there cannot be a sequential equilibrium where \( T \) is played with probability 1 because of the following reasons. (1) \( T \) is better than \( B \) if and only if player 2 plays \( l \) with sufficiently high probability; (2) player 2 plays \( l \) with sufficiently high probability if and only if she attaches high probability to being at node \( x_5 \); (3) positive probability of being at node \( x_5 \) is allowed only if player 1 plays \( R \) and \( b \) with positive probability (otherwise, if he plays \( L \) with probability 1, reaching \( x_5 \) requires two mistakes: \( M \) and \( R \), while reaching \( x_4 \) requires only the mistake \( M \); two mistakes are infinitely less likely than one of the two mistakes only); (4) but if \( l \) is player with sufficiently high probability, then player 1 will in fact want to choose \( L \) at \( x_1 \).