so that \( E_t p_{t+1} = p_0 + \mu p(t+1) \). Substituting these into (72),

\[
p_t = \frac{b \mu t + r - bc + p_0 + \mu p(t+1)}{1 + b} = \frac{r - bc + p_0 + \mu p}{1 + b} + \left[ \frac{b \mu + \mu p}{1 + b} \right] t,
\]

which equals \( p_0 + \mu p_t \) if and only if

\[
\mu p = \left[ \frac{b \mu + \mu p}{1 + b} \right] \Rightarrow \mu = \mu
\]

and

\[
p_0 = \frac{r - bc + p_0 + \mu p}{1 + b} = \frac{r - bc + \mu p}{1 + b} = \frac{r - bc + \mu}{b}.
\]

With these expression for \( p_0 \) and \( \mu p \), real money balances will equal

\[
m_t - p_t = \mu t - \left( \frac{r - bc + \mu}{b} + \mu t \right) = - \left( \frac{r - bc + \mu}{b} \right),
\]

which is decreasing in the growth rate of nominal money balances \( \mu \). From the solution for the price level, the rate of inflation is equal to \( \mu \). Higher values of \( \mu \), and therefore higher rates of inflation, increase the opportunity cost of holding money. This reduces the real demand for money, and, in equilibrium, real money balances are lower at higher rates of money growth.

7. Consider a simple forward-looking model of the form

\[
x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) + u_t,
\]

\[
\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + e_t.
\]

Suppose policy reacts to the output gap:

\[
i_t = \delta x_t.
\]

Write this system in the form given by (5.71). Are there values of \( \delta \) that ensure a unique stationary equilibrium? Are there values that do not?

Using the policy rule to eliminate \( i_t \), the system can be written as

\[
\begin{bmatrix}
1 & \sigma^{-1} \\
0 & \beta
\end{bmatrix}
\begin{bmatrix}
E_t x_{t+1} \\
E_t \pi_{t+1}
\end{bmatrix}
= \begin{bmatrix}
1 + \sigma^{-1} \delta & 0 \\
-\kappa & 1
\end{bmatrix}
\begin{bmatrix}
x_t \\
\pi_t
\end{bmatrix}
- \begin{bmatrix}
u_t \\
e_t
\end{bmatrix}.
\]

Premultiplying both sides by

\[
\begin{bmatrix}
1 & \sigma^{-1} \\
0 & \beta
\end{bmatrix}^{-1} = \begin{bmatrix}
\frac{1}{\beta} & -\sigma^{-1} \\
0 & 1
\end{bmatrix}
\]

50
yields
\[
\begin{bmatrix}
E_t x_{t+1} \\
E_t \pi_{t+1}
\end{bmatrix} = M \begin{bmatrix}
x_t \\
\pi_t
\end{bmatrix} - \left( \frac{1}{\beta} \right) \begin{bmatrix}
\beta u_t + \sigma^{-1} e_t \\
e_t
\end{bmatrix},
\]
where
\[
M = \left( \frac{1}{\beta} \right) \begin{bmatrix}
\beta & -\sigma^{-1} & 1 + \sigma^{-1} \delta & 0 \\
0 & 1 & -\kappa & 1 \\
\beta (1 + \sigma^{-1} \delta) + \sigma^{-1} \kappa & -\sigma^{-1} \kappa & -\kappa & 1
\end{bmatrix}.
\]

The characteristic polynomial of \( M \) is defined as \( f(\lambda) = \det(M - \lambda I) \), where \( I \) is the 4x4 identity matrix. Thus,
\[
f(\lambda) = \begin{vmatrix}
1 + \frac{\delta}{\sigma} + \frac{\kappa}{\beta \sigma} - \lambda & -\frac{1}{\beta \sigma} \\
-\frac{\kappa}{\beta} & \frac{1}{\sigma} - \lambda
\end{vmatrix}
= \lambda^2 - \left( \frac{1}{\beta \sigma} \right) \left[ \sigma (1 + \beta) + \beta \delta + \kappa \right] \lambda - \frac{\kappa}{\beta \sigma}
= \lambda^2 - a_1 \lambda - a_0,
\]
where \( a_1 \) and \( a_0 \) are both non-negative if we assume \( \delta > 0 \). Determinacy requires that both eigenvalues be outside the unit circle. The eigenvalues of \( M \), \( \lambda_1 \) and \( \lambda_2 \), satisfy \( f(\lambda_i) = 0 \). The model of this problem corresponds to the baseline model of Bullard and Mitra (2002) when policy reacts only to the output gap, although they write the model as
\[
\begin{bmatrix}
x_t \\
\pi_t
\end{bmatrix} = N \begin{bmatrix}
E_t x_{t+1} \\
E_t \pi_{t+1}
\end{bmatrix} + \begin{bmatrix}
u_t \\
e_t
\end{bmatrix}
\]
and so require the eigenvalues of \( N \) to be inside the unit circle. They show that this will be the case if and only if
\[
(1 - \beta) \delta - \kappa > 0.
\]

Hence, the rational expectations solution is unique if the central bank responds sufficiently strongly to the output gap (i.e., as long as \( \delta > \kappa/(1 - \beta) > 0 \)).

8. Consider the model given by
\[
x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1})
\]
\[
\pi_t = \beta E_t \pi_{t+1} + \kappa x_t.
\]
Suppose policy sets the nominal interest rate according to a policy rule of the form
\[
i_t = \phi_1 E_t \pi_{t+1}
\]
for the nominal rate of interest.
(a) Write this system in the form $E_t z_{t+1} = M z_t + \eta_t$, where $z_t = [x_t, \pi_t]$. Using the policy rule to eliminate $i_t$ from the IS equation, the system can be written as

$$
\begin{bmatrix}
1 & \sigma^{-1}(1-\phi_1) \\
0 & \beta
\end{bmatrix}
\begin{bmatrix}
E_t x_{t+1} \\
E_t \pi_{t+1}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
-\kappa & 1
\end{bmatrix}
\begin{bmatrix}
x_t \\
\pi_t
\end{bmatrix}.
$$

Premultiplying both sides by

$$
\begin{bmatrix}
1 & \sigma^{-1}(1-\phi_1) \\
0 & \beta
\end{bmatrix}^{-1}
= \left(\frac{1}{\beta}\right)
\begin{bmatrix}
\beta & -\sigma^{-1}(1-\phi_1) \\
0 & 1
\end{bmatrix}
$$

yields

$$
\begin{bmatrix}
E_t x_{t+1} \\
E_t \pi_{t+1}
\end{bmatrix}
= M
\begin{bmatrix}
x_t \\
\pi_t
\end{bmatrix},
$$

where

$$
M
= \left(\frac{1}{\beta}\right)
\begin{bmatrix}
\beta & -\sigma^{-1}(1-\phi_1) \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-\kappa & 1
\end{bmatrix}
= \left(\frac{1}{\beta}\right)
\begin{bmatrix}
\beta + \sigma^{-1}(1-\phi_1)\kappa & -\sigma^{-1}(1-\phi_1) \\
-\kappa & 1
\end{bmatrix}.
$$

Note that there were no disturbance terms included in the model, so $\eta \equiv 0$.

(b) For $\beta = 0.99$, $\kappa = 0.05$, and $\sigma = 1.5$, plot the absolute values of the two eigenvalues of $M$ as a function of $\phi_1 > 0$. The eigenvalues are the solutions to

$$
\begin{vmatrix}
1 + \frac{\frac{\sigma^{-1}(1-\phi_1)}{\beta} \kappa}{\frac{1}{\beta}} & -\frac{\sigma^{-1}(1-\phi_1)}{\beta} \\
-\frac{\sigma^{-1}(1-\phi_1)}{\beta} & \frac{1}{\beta} - \lambda
\end{vmatrix} = 0.
$$

Evaluating the determinant yields

$$
\lambda^2 - \text{trace}(M) \lambda - \det(M) = 0,
$$

where

$$
\text{trace}(M) = 1 + \frac{\sigma^{-1}(1-\phi_1)\kappa}{\beta} + \frac{1}{\beta}.
$$

Using the given values for $\beta$, $\kappa$, and $\sigma$, this becomes

$$
\lambda^2 - [2.02438 - 0.0337\phi_1] \lambda - 0.034 + 0.034\phi_1 = 0.
$$

Proposition 4 and Appendix D of Bullard and Mitra (2002) apply to this case, and they show that the equilibrium is unique if

$$
0 < \kappa(\phi_1 - 1) < 2\sigma(1 + \beta).
$$
Figure 5: Ch. 5, Problem 8: Minimum Eigenvalue ($0 < \phi_1 < 2$)

Figure 6: Ch. 5, Problem 8: Minimum Eigenvalue ($110 < \phi_1 < 130$)
With the parameter values given in this problem, this requires that

\[ 1 < \phi_1 < \frac{2\sigma(1 + \beta)}{\kappa} + 1 = 120.4. \]

It turns out to be more illustrative to plot separately the minimum eigenvalue for \( \phi_1 \) near 1 and near 120. This is done in the figures (5) and (6).

(c) Are there values of \( \phi_1 \) for which the economy does not have a unique stationary equilibrium? Yes. If \( \phi_1 < 1 \) or \( \phi_1 > 120.4 \), the rational expectations equilibrium is not unique because both eigenvalues of \( M \) are not outside the unit circle.

9. Suppose the economy is described by the basic new Keynesian model consisting of

\[
\begin{align*}
    x_t &= E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) \\
    \pi_t &= \beta E_t \pi_{t+1} + \kappa x_t \\
    i_t &= \phi_\pi \pi_t + \phi_x x_t.
\end{align*}
\]

(a) If \( \phi_x = 0 \), explain intuitively why \( \phi_\pi > 1 \) is needed to ensure that the equilibrium will be unique. If private agents were to expect higher inflation, current inflation would rise (from the inflation adjustment relationship). If \( 0 < \phi_\pi < 1 \), the rise in inflation induces a policy response that leads to a rise in the nominal rate of interest, but the increase in \( i_t \) is not sufficient to raise the real rate of interest. A decline in the real rate causes current output to rise, pushing up actual inflation and validating the initial rise in expected inflation. Thus, multiple rational expectations equilibrium are possible. If \( \phi_\pi > 1 \), then the rise in inflation induces a policy response that is strong enough to raise the real rate of interest, reduce the output gap, and lower actual inflation. This ensures a unique equilibrium.

(b) If both \( \phi_\pi \) and \( \phi_x \) are nonnegative, the condition given by (5.73) implies that the economy can still have unique, stable equilibrium even when

\[ 1 - \frac{(1 - \beta)\phi_x}{\kappa} < \phi_\pi < 1. \]

Explain intuitively why some values of \( \phi_\pi < 1 \) are still consistent with stability when \( \phi_x > 0 \). The nominal rate is now affected by both inflation and the output gap. As explained under part a, if \( \phi_\pi < 1 \), the real rate would fall, acting to increase the output gap. However, if \( \phi_x \) is large enough, this rise in the gap induces a strong enough nominal interest rate increase to ensure a rise in expected inflation is not self-fulfilling.