TOPIC 5. THE TERM STRUCTURE OF INTEREST RATES
• Aggregate spending decisions are generally viewed as more closely related to long-term interest rates.

• However, the short-term interest rate is the operational target for implementing monetary policy. Hence, aggregate spending decisions depend on the link between short-term and long-term rates.

• **Expectations Hypothesis of the Term Structure:** long-term nominal interest rates depend on expectations of future nominal short-term interest rates. Hence, expectations about future policy play a big role.
Preliminaries – Bond Pricing

General View

\[ p_t = E(m_{t+1} x_{t+1}) \]
\[ m_{t+1} = f(data, parameters) \]

- \( p_t \) is the asset price
- \( x_{t+1} \) is the asset payoff
- \( m_{t+1} \) is the stochastic discount factor or pricing kernel
General Principles

• the marginal utility loss of consuming a little less today = the marginal utility gain of consuming a little more of the asset’s pay-off tomorrow.

• risk depends on the correlation between the asset’s payoffs and the marginal utility of future consumption. Prices will hence reflect a discount for “riskiness.”
Basic Asset-Pricing Model

\[ u(c_t, c_{t+1}) = u(c_t) + \beta E_t [u(c_{t+1})] \]

\[ \max u(c_t) + \beta E_t [u(c_{t+1})] \quad \text{s.t.} \]

\[ c_t = e_t - p_t \gamma \]

\[ c_{t+1} = e_{t+1} + x_{t+1} \gamma \]

\( e_t \) is the consumption level without the asset

\( \gamma \) is the amount of the asset purchased
F.O.C.

\[ p_t = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right] \]

It is convenient to define

\[ m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \text{ s.t. } p_t = E_t \left( m_{t+1} x_{t+1} \right) \]
The risk-free rate is related to the discount factor by

\[ R_f = \frac{1}{E(m)} \]

In the pricing formula, let \( p_t = 1; x_{t+1} = R_f \), hence

\[ 1 = E_t \left( m_{t+1} R_f \right) \implies R_f = \frac{1}{E_t \left( m_{t+1} \right)} \]
Risk corrections

\[ E(x, y) = \text{cov}(x, y) + E(x)E(y) \]

\[ p_t = E_t(m_{t+1}x_{t+1}) = \text{cov}_t(m_{t+1}x_{t+1}) + E_t(m_{t+1})E_t(x_{t+1}) \]

\[ p_t = \frac{E_t(x_{t+1})}{R^f} + \text{cov} (m_{t+1} x_{t+1}) \]
A simple discrete-time model of the term structure

Assume the pricing kernel follows an AR(1). Let $E(\ln m) = -\delta$, $\delta > 0$. Then

$$\ln m_{t+1} + \delta = \rho \{ \ln m_t + \delta \} + \varepsilon_{t+1}$$

The key will be to link a model for $m_{t+1}$ with a macro model so that the dynamics of $m_{t+1}$ are a function of macro-variables.
Let $\ln P_t^{(N)} = p_t^{(N)}$. For example, if the price of a one-year, zero-coupon bond is 0.95, then $\ln 0.95 = -0.051$, i.e., the bond sells at a 5% discount.

Also, logs give a continuously compounded rate. If $e^{rN} = \frac{1}{P^{(N)}}$ then $rN = -\ln P^{(N)} = -p^{(N)}$

The yield is $P^{(N)} = \frac{1}{(Y^{(N)})^N}$ or $y^{(N)} = -\frac{1}{N} p^{(N)}$
Using the standard asset pricing formula

\[-p_t^{(1)} = y_t^{(1)} = -\ln E_t \left( e^{\ln m_{t+1}} \right)\]

\[-\frac{1}{2} p_t^{(2)} = y_t^{(2)} = -\frac{1}{2} \ln E_t \left( e^{\ln m_{t+1} + \ln m_{t+2}} \right)\]

\[\vdots\]

Given the assumption on the pricing kernel and iterating forward

\[\ln m_{t+2} + \delta = \rho^2 \left( \ln m_t + \delta \right) + \rho \varepsilon_{t+1} + \varepsilon_{t+2}\]

\[\vdots\]
hence

\[ \ln(m_{t+1} + \delta) + \ln(m_{t+2} + \delta) = (\rho + \rho^2)(\ln m_t + \delta) + (1 + \rho)\varepsilon_{t+1} + \varepsilon_{t+2} \]

the log-normal rule establishes that

\[ E(e^x) = e^{E(x) + \frac{1}{2}\sigma_x^2} \]

hence

\[ y_t^{(1)} = \delta - \rho(\ln m_t + \delta) - \frac{1}{2} \sigma_{\varepsilon}^2 \]
\[ y_t^{(2)} = \delta - \frac{\rho + \rho^2}{2} \left[ y_t^{(1)} - E(y^{(1)}) \right] - \frac{1 + (1 + \rho)^2}{4} \sigma_{\varepsilon}^2 \]
\[ y_t^{(3)} = \delta - \frac{\rho + \rho^2 + \rho^3}{3} \left[ y_t^{(1)} - E(y^{(1)}) \right] - \frac{1 + (1 + \rho)^2 + (1 + \rho + \rho^2)^2}{6} \sigma_{\varepsilon}^2 \]

\[ \vdots \]
The Expectations Hypothesis

The $n$-period interest rate will be equal to the average of the current, one-period short-term rate and the future one-period, short-term rates expected to hold over the $n$-period horizon.

Without uncertainty or risk premia:

$$(1 + i_{n,t})^n = \prod_{i=0}^{n-1} (1 + i_{t+i})$$
Taking logs

\[ i_{n,t} = \frac{1}{n} \sum_{i=0}^{n-1} i_{t+i} \]

Similarly

\[ i_{n,t} = \frac{1}{n} i_t + \frac{n-1}{n} i_{n-1,t+1} \]
Typical Empirical Specifications

\[ I_t = \frac{1}{2} (i_t + E_t i_{t+1}) \Rightarrow I_t - i_t = \frac{1}{2} (E_t i_{t+1} - i_t) \]

Since

\[ i_{t+1} = E_t i_{t+1} + (i_{t+1} - E_t i_{t+1}) \]

then

\[ \frac{1}{2} (i_{t+1} - i_t) = I_t - i_t + \frac{1}{2} (i_{t+1} - E_t i_{t+1}) \]

\[ = a + b (I_t - i_t) + \theta_{t+1} \]
where $a = 0$, $b = 1$, and $\theta_{t+1} = \frac{1}{2}(i_{t+1} - E_t i_{t+1})$ is the forecast error, hence uncorrelated with time $t$ information.

*However, there is NO empirical support for this relations!!*
Principal Components

Let $\mathbf{X} \in \mathbb{R}^{T \times n}$ with $E(\mathbf{X}' \mathbf{X}) = \Sigma$, $E(\mathbf{X}) = 0$

principal components consist in taking the linear combinations of $X$ that have the maximum variance and are uncorrelated to each other, i.e.

$$\max_{\beta} \frac{1}{T} \beta' \mathbf{X}' \mathbf{X} \beta - \lambda (\beta' \beta - I)$$
F.O.C.

\[ \left( \frac{X'X}{T} - \lambda I \right) \beta = 0 \implies \left| \frac{X'X}{T} - \lambda I \right| = 0 \]

the eigenvalues of \( \frac{X'X}{T} \) ordered from largest to smallest are

\[ \beta' \frac{X'X}{T} \beta = \lambda \]

and the corresponding eigenvectors are the \( \beta \)'s.

Hence, \( u = X\beta \) is such that \( E(u) = 0 \) and \( E(u'u) = \Lambda \)