0.1 Theoretical Questions

1. Find the mean and the autocovariance function of the process

\[ x_t = 2 + 1.3x_{t-1} - 0.4x_{t-2} + z_t + z_{t-1} \quad \{z_t\} \sim WN(0, \sigma^2) \]

Solution:

- Start by checking stationarity, i.e. the roots of \((1 - 1.3x + 0.4x^2) = 0\). These are \(x = 2, 1.25\), clearly outside the unit circle so the process is stationary. Therefore, \((1 - 1.3L + 0.4L^2) = (1 - 0.5L)(1 - 0.8L)\). Now that we know the process is stationary, we can calculate the mean,

\[
E(x_t) = 2 + 1.3E(x_t) - 0.4E(x_{t-2})
\]

Stationarity means, \(E(x_t) = E(x_{t-j})\), therefore, \(E(x_t) = 2/(1 + 1.3 - 0.4) = 20\).

- The autocovariance function can be easily calculated using the Yule-Walker equations since \(x_t\) is stationary

\[
\begin{align*}
\gamma_0 &= 1.3\gamma_1 - 0.4\gamma_2 + 2\sigma^2 \\
\gamma_1 &= 1.3\gamma_0 - 0.4\gamma_1 + \sigma^2 \\
\gamma_2 &= 1.3\gamma_1 - 0.4\gamma_0 \\
&\vdots \\
\gamma_r &= 1.3\gamma_{r-1} - 0.4\gamma_{r-2} \quad \text{for } r \geq 3
\end{align*}
\]

1. (a) Is this process covariance-stationary? Yes
(b) Is this process invertible? No since the MA polynomial $(1 + L)$ is not invertible.

2. Find the coefficients $\psi_j, j = 0, 1, 2, \ldots$ in the representation

$$x_t = \sum_{j=0}^{\infty} \psi_j z_{t-j}$$

of the ARMA(2,1) process

$$(1 - 0.5L + 0.04L^2)x_t = (1 + 0.25L)z_t \quad \{z_t\} \sim WN(0, \sigma^2)$$

Solution:

- First check stationarity.

$$(1 - 0.5L + 0.04L^2) = (1 - 0.1L)(1 - 0.4L) \text{ so the process is covariance stationary. Hence}
\begin{align*}
  x_t &= \frac{(1 + 0.25L)}{(1 - 0.1L)(1 - 0.4L)} z_t \\
  \text{hence,}
  x_t &= (1 + 0.25L)(1 + 0.1L + 0.1^2L^2 + \ldots)(1 + 0.4L + 0.4^2L^2 + \ldots)z_t
\end{align*}$$

tedious algebra would deliver the $\psi_j$'s requested.

1. (a) How would you find said coefficients computationally, say, using EXCEL? Let cells $A1 = A2 = B1 = B2 = 0$ and let cell $B3 = 1$. Then type the following formula in cell $A3$

$$= 0.5 \times A2 - 0.04 \times A1 + B3 + 0.25 \times B2$$

Copy cell $A3$ down the $A$ column for as many $\psi_j$'s you want to compute. The first few $\psi_j$ are, $\psi_1 = 0.75; \psi_2 = 0.335; \psi_3 = 0.1375; \psi_4 = 0.05535; \text{ and so on.}$

3. Let $\{y_t\}$ be a stationary, zero-mean, time series. Define

$$x_t = (1 - 0.4L)y_t$$

$$w_t = (1 - 2.5L)y_t$$
(a) Express the autocovariance functions of \( \{x_t\} \) and \( \{w_t\} \) in terms of the autocorrelation function of \( \{y_t\} \).

\[
\begin{align*}
\gamma_x^0 &= 1.16\gamma_y^0 - 0.8\gamma_y^1 \\
\gamma_x^1 &= 1.16\gamma_y^1 - 0.4\gamma_y^0 - 0.4\gamma_y^2 \\
&\vdots \\
\gamma_x^r &= 1.16\gamma_y^r - 0.4\gamma_y^{r-1} - 0.4\gamma_y^{r+1} \quad r \geq 2
\end{align*}
\]

\[
\begin{align*}
\gamma_w^0 &= 7.25\gamma_y^0 - 5\gamma_y^1 \\
\gamma_w^1 &= 7.25\gamma_y^1 - 2.5\gamma_y^0 - 2.5\gamma_y^2 \\
&\vdots \\
\gamma_w^r &= 7.25\gamma_y^r - 2.5\gamma_y^{r-1} - 2.5\gamma_y^{r+1} \quad r \geq 2
\end{align*}
\]

(b) Show that \( \{x_t\} \) and \( \{w_t\} \) have the same autocorrelation function.

\[
\begin{align*}
\rho_x^1 &= \frac{\gamma_x^1}{\gamma_x^0} = \frac{(1.16\rho_y^1 - 0.4 - 0.4\rho_y^2)\gamma_y^0}{(1.16 - 0.8\rho_y^1)\gamma_y^0} \\
&\vdots \\
\rho_x^r &= \frac{(1.16\rho_y^r - 0.4\rho_y^{r-1} - 0.4\rho_y^{r+1})}{(1.16 - 0.8\rho_y^1)} \quad r \geq 2 \\
\rho_w^1 &= \frac{(7.25\rho_y^1 - 2.5 - 2.5\rho_y^2)}{(7.25 - 5\rho_y^1)} = \frac{6.25(1.16\rho_y^1 - 0.4 - 0.4\rho_y^2)}{6.25(1.16 - 0.8\rho_y^1)} = \rho_x^1
\end{align*}
\]

similarly for \( r \geq 2 \).

(c) Show that the process \( u_t = -\sum_{j=1}^{\infty}(0.4)^jx_{t+j} \) satisfies the difference equation \( u_t - 2.5u_{t-1} = x_t \).

\[
\begin{align*}
u_t &= -\sum_{j=1}^{\infty}(0.4)^jx_{t+j} = x_t - \sum_{j=0}^{\infty}(0.4L^{-1})^jx_t; \\
u_t &= x_t - \frac{1}{1 - 0.4L^{-1}}x_t; \\
(1 - 0.4L^{-1})u_t &= -0.4L^{-1}x_t; \\
u_t &= x_t - 2.5u_{t-1} \quad Q.E.D.
\end{align*}
\]

4. The series \( \{x_t\} \) is generated by the process \( x_t + \alpha x_{t-1} = \varepsilon_t + \beta \varepsilon_{t-1} \) where \(-1 < \alpha, \beta < 1\).

(a) Show that stationarity holds under the appropriate initial conditions. Stationarity holds since \( |\alpha| < 1 \) and as long as \( |x_0| < \infty \), and \( \varepsilon_0 \sim WN(0,\sigma^2) \) with \( \sigma^2 < \infty \).
(b) Let $V(x_t)$ denote the variance of $x_t$, establish that under stationarity

$$V(x_t) = \sigma^2 \frac{(1 - 2\alpha\beta + \beta^2)}{(1 - \alpha^2)}$$

and

$$\rho_1 = \frac{(\beta - \alpha)(1 - \alpha\beta)}{(1 - 2\alpha\beta + \beta^2)}$$

with $\rho_r = \alpha \rho_{r-1}$ for $r = 2, 3, \ldots$ where $\rho_r = E(x_t x_{t-r})/E(x_t^2)$.

**Solution:**

$$x_t = \frac{1}{1 + \alpha L} \varepsilon_t + \frac{\beta}{1 + \alpha L} \varepsilon_{t-1}$$

hence

$$E(x_t^2) = E \left( \left\{ \varepsilon_t - \alpha \varepsilon_{t-1} + \alpha^2 \varepsilon_{t-2} - \ldots \right\}^2 + \beta \left[ \varepsilon_{t-1} - \alpha \varepsilon_{t-2} + \alpha^2 \varepsilon_{t-3} - \ldots \right] \right)^2 =$$

$$= \frac{1}{1 - \alpha^2} \sigma^2 + \frac{\beta^2}{1 - \alpha^2} \sigma^2 - \frac{2\alpha}{1 - \alpha^2} \sigma^2$$

$$= \frac{1 + \beta^2 - 2\alpha\beta}{1 - \alpha^2} \sigma^2$$

Q.E.D.

Similarly, for $\rho_1$ we can now use the Yule-Walker equations to obtain

$$E(x_t x_{t-1}) = -\alpha E(x_t^2) + \beta \sigma^2$$

so that

$$\rho_1 = -\alpha + \frac{\beta}{E(x_t^2)} \sigma^2 = \frac{\beta(1 - \alpha^2) \sigma^2}{(1 - 2\alpha\beta + \beta^2) \sigma^2} - \alpha$$

$$= \frac{(\beta - \alpha)(1 - \alpha\beta)}{(1 - 2\alpha\beta + \beta^2)}$$

Q.E.D.

and $\rho_r = -\alpha \rho_{r-1}$, for $r \geq 2$.

(c) Derive the values of $\alpha$ and $\beta$ from knowledge of $\rho_r$, $r = 2, 3, \ldots$

**Solution:**

$\alpha = -\rho_r / \rho_{r-1}$. Note $\beta$ is the MA parameter so it can only be identified from $\rho_1$ once we obtain the value for $\alpha$.

5. Let $\{\varepsilon_t\}$ be a white noise. Verify that the process defined as $x_t = \varepsilon_t$ and $y_t = (-1)^t \varepsilon_t$ are stationary. Show that their sum, $z_t = x_t + y_t$ is not stationary.
Solution:

$E(x_t) = 0$, $V(x_t) = \sigma^2$, therefore, $x$ is covariance-stationary. Similarly, $E(y_t) = 0$, $V(y_t) = \sigma^2$, also covariance-stationary. However, $E(z_t) = 0$ but $V(z_t) = 0$ if $t$ odd, $2\sigma^2$ if $t$ even, so that the variance depends on $t$ and thus the process is not stationary.

6. Consider an AR(1) process following

$$y_t = c + \phi y_{t-1} + \varepsilon_t$$

where the process started at $t = 0$ with the initial value $y_0$. Suppose $|\phi| < 1$ and that $y_0$ is uncorrelated with the subsequent white noise process $(\varepsilon_1, \varepsilon_2, ...)$.

(a) Write the variance of $y_t$ as a function of $\sigma^2$ ($\equiv V(\varepsilon_t)$); $V(y_0)$, and $\phi$. Hint: By successive substitution, show

$$y_t = (1 + \phi + \phi^2 + ... + \phi^{t-1})c + \varepsilon_t + \phi \varepsilon_{t-1} + ... + \phi^{t-1} \varepsilon_1 + \phi^t y_0$$

Solution:

Using the above formula,

$$V(y_t) = (1 + \phi^2 + \phi^4 + ... + \phi^{2(t-1)}) \sigma^2 + \phi^{2t} V(y_0)$$

$$= \frac{1 - \phi^{2t}}{1 - \phi^2} \sigma^2 + \phi^{2t} V(y_0)$$

(b) Let $\mu$ and $\{\gamma_j\}$ be the mean and the autocovariances of the covariance-stationary AR(1) process, so $\mu = c/(1-\phi)$ and $\gamma_j = \phi^j \sigma^2/(1 - \phi^2)$. Show that:

$$\lim_{t \to \infty} E(y_t) = \mu$$

$$\lim_{t \to \infty} V(y_t) = \gamma_0$$

$$\lim_{t \to \infty} COV(y_t, y_{t-j}) = \gamma_j$$

Solution:
\[
\lim_{t \to \infty} E(y_t) = \lim_{t \to \infty} E\left\{ \frac{1 - \phi^t}{1 - \phi} \right\} = \frac{1}{1 - \phi} c
\]
\[
\lim_{t \to \infty} V(y_t) = \lim_{t \to \infty} \left\{ \frac{1 - \phi^{2t}}{1 - \phi^2} \right\} = \frac{1}{1 - \phi^2} \sigma^2
\]
\[
\lim_{t \to \infty} COV(y_t y_{t-j}) = \lim_{t \to \infty} \left\{ \phi^j \frac{(1 - \phi^{2t})}{1 - \phi^2} \right\} = \frac{\phi^j}{1 - \phi^2} \sigma^2
\]

### 0.2 Applied Questions

1. The models for question 3 are

   \[
   x_t = 2 + 0.5x_{t-1} + \varepsilon_t + 0.7\varepsilon_{t-1}
   \]

   \[
   y_t = 1 + 0.5y_{t-1} - 0.2y_{t-2} + \varepsilon_t
   \]