PROBLEM SET 1 – SOLUTIONS

Part I – Analytical Questions

1. Suppose

\[ \sqrt{T} \left( \hat{\theta}_T - \theta_0 \right) \xrightarrow{L} N(0, \sigma^2) \]

Does it follow then that \( \hat{\theta}_T \xrightarrow{p} \theta_0 \)? Explain your answer. Hint:

\[ \hat{\theta}_T - \theta_0 = \frac{1}{\sqrt{T}} \sqrt{T} \left( \hat{\theta}_T - \theta \right) \text{ and } p \lim_{T \to \infty} \frac{1}{\sqrt{T}} = 0 \]

Solution

We know the result holds as long as \( \theta_0 \) is a constant. This is true in this case since \( \theta_0 \) is the population mean. Also, notice that since \( p \lim_{T \to \infty} \frac{1}{\sqrt{T}} = 0 \), and

\[ \sqrt{T} \left( \hat{\theta}_T - \theta_0 \right) \xrightarrow{L} N(0, \sigma^2) \], which is therefore bounded (or \( O_p(T^{1/2}) \)), it immediately follows that

\[ p \lim_{T \to \infty} \hat{\theta}_T - \theta_0 = 0 \]

2. Combining the delta method and Lindeberg-Levy.

Let \( \{z_t\} \) be an i.i.d. sequence of random variables with \( E(z_t) = \mu \neq 0 \) and \( V(z_t) = \sigma^2 \). Let \( \bar{z}_T \) be the sample mean, derive the asymptotic distribution of \( \frac{1}{\bar{z}_T} \).

Solution

By Lindeberg-Levy and the conditions of the problem it is easy to see that

\[ \sqrt{T} \left( \bar{z}_T - \mu \right) \xrightarrow{L} N(0, \sigma^2) \]

Next, apply the delta method. First, note that \( g(x) = 1/x \), that is, in this case we are taking the inverse transform, and therefore, \( g'(x_0) = -1/(x_0)^2 \). Here, \( x = \bar{z}_T \) and \( x_0 = \mu \). Therefore

\[ \sqrt{T} \left( \frac{1}{\bar{z}_T} - \frac{1}{\mu} \right) \xrightarrow{L} N \left( 0, \frac{\sigma^2}{\mu^4} \right) \]
3. Show that a random walk is not stationary (hint: check the variance).

Solution

Consider a simple random walk, such as

\[ y_t = y_{t-1} + \epsilon_t \]

with \( V(\epsilon) = \sigma^2 \). Then, notice that each element in the sequence \( \{y_t\} \) is just the sum of all past innovations, i.e.

\[ y_t = \sum_{s=0}^{t} \epsilon_s \]

and therefore \( V(y_t) = \sum_{s=0}^{t} \sigma^2 = t\sigma^2 \) (by the independence of the \( \epsilon \)). Hence, as \( t \) grows so does the variance of \( y \).

4. Suppose \( X_T \xrightarrow{p} c \) and \( Y_T \xrightarrow{p} Y \) where \( Y \sim N(\mu, \sigma^2) \). Derive the limiting distribution of \( XTYT \). Be explicit about your assumptions.

Solution

We know that convergence in probability implies convergence in distribution. Also from Slutsky’s theorem we know that if \( X_T \xrightarrow{p} c \) and \( Y_T \xrightarrow{L} Y \) then \( X_T Y_T \xrightarrow{L} cY \).

Since \( Y \) is normally distributed, then \( cY \sim N(c\mu, c^2\sigma^2) \).

5. Does a martingale difference sequence have to be covariance stationary? Explain.

Solution

No, the variance could be a function of time.

6. Find the orders \( o(.) \) and \( O(.) \) of the following sequences:

- \( b_T = 4 + 2T + 6T^2 \). Solution: \( O(T^2) \); \( o(T^{2+\delta}) \); \( \delta > 0 \)
- \( b_T = (-1)^T \). Solution: \( O(1) \); \( o(T^{\delta}) \); \( \delta > 0 \)
- \( b_T = \exp(-T) \). Solution: \( O(T^{\delta}) \); \( o(T^{\delta}) \); \( \delta > 0 \)
- \( b_T = \exp(T) \). Solution: unbounded!
7. Let \( \{x_t\} \) be a sequence of random variables such that \( E(x_t) = \mu \), and \( V(x_t) = \sigma^2 \) for all \( t \), and \( COV(z_t, z_\tau) = 0 \) for all \( t \neq \tau \). Show that:

\[
\bar{X}_T \xrightarrow{p} \mu
\]

Solution:
Notice that the \( x_t \) are not assumed to be i.i.d. (they are simply uncorrelated). Therefore, we cannot directly apply the LLNs that we have studied in class.

\[
\bar{X}_T = \frac{1}{T} \sum_{t=1}^{T} x_t; \text{ hence } E(\bar{X}_T) = \mu \text{ from the statement of the problem. Next, we need to show that } E\left( \frac{1}{T} \sum_{t=1}^{T} x_t - \frac{T}{T} \mu \right)^2 = 0.
\]

\[
E\left( \frac{1}{T} \sum_{t=1}^{T} x_t - \frac{T}{T} \mu \right)^2 = E\left( \frac{1}{T} \sum_{t=1}^{T} (x_t - \mu) \right)^2 = \frac{1}{T^2} E\left( [x_1 - \mu] + [x_2 - \mu] + \ldots + [x_T - \mu] \right)^2 = \frac{1}{T^2} E\left( [x_1 - \mu] \times [x_1 - \mu] + \ldots + [x_T - \mu] \times [x_T - \mu] \right)
\]

\[
= \frac{1}{T^2} T\sigma^2 \text{ since all the covariance terms are zero by assumption. Therefore, we have shown that } E\left( \frac{1}{T} \sum_{t=1}^{T} x_t - \frac{T}{T} \mu \right)^2 \rightarrow 0 \text{ as } T \rightarrow \infty, \text{ which implies that } \bar{X}_T \xrightarrow{m.s.} \mu \text{. Finally, in class we discussed that mean square convergence implies convergence in probability, thus completing the proof.}
\]

8. Let \( \{u_t\} \) be a sequence of i.i.d. random variables uniformly distributed on \([0, 1]\) and let the random variable \( X \sim N(0, 1) \) and independent for all \( t \). Let \( Y_t = X + u_t \) (note \( X \) has no subscript \( t \)). Show that:

- \( \{Y_t\} \) is stationary
- \( \bar{Y}_T \) does not converge in probability to \( \frac{1}{2} \)
- \( \bar{Y}_T - X \xrightarrow{s.a.} 1/2 \)

Solution:
Notice \( E(Y_t) = \frac{1}{2} \) and \( E(Y_t - \frac{1}{2})^2 = 1/4 \) for all \( t \). Finally, \( COV(Y_t, Y_{t'}) = V(X) = 1 \) for all \( t \). This concludes the stationarity proof. Next, notice that, because \( X \) is common to all \( Y_t \) the dependence structure never dies out and in fact, \( \bar{Y}_T \) does not converge to a constant but to a random variable, \( \frac{1}{2} + X \). Hence the difference \( \bar{Y}_T - X \) will indeed converge to \( \frac{1}{2} \).
9. Let \( Y_t = \sum_{j=0}^{\infty} \phi_j \varepsilon_{t-j} \) where \( \sum_{j=0}^{\infty} |\phi_j| < \infty \) and \( \{\varepsilon_t\} \) is a martingale difference sequence with \( E(\varepsilon_t^2) = \sigma^2 \). Is \( Y_t \) covariance stationary? Justify your answer.

Solution

Yes, \( \varepsilon_t \) has variance \( \sigma^2 \) for all \( t \). Since \( \varepsilon_t \) is a m.d.s., it has mean zero and must be serially uncorrelated. Thus \( \{\varepsilon_t\} \) is white noise and this is a covariance-stationary \( MA(\infty) \) process.

10. Let \( Y_t \) follow the process

\[
(1 - \phi_1 L - \ldots - \phi_p L^p)(Y_t - \mu) = (1 + \theta_1 L + \ldots + \theta_q L^q)\varepsilon_t
\]

with the roots of the AR and MA polynomials outside the unit circle (i.e. the process is stationary and invertible). Suppose \( E(\varepsilon_t) = 0 \) for all \( t \), uncorrelated and \( E(\varepsilon_t^2) = \sigma^2 \) and \( E(\varepsilon_t^4) < \infty \). Prove:

\[
\frac{1}{T} \sum_{t=1}^{T} Y_t \overset{p}{\longrightarrow} \mu
\]

Solution:

By the conditions of the problem we know we can write

\[
Y_t = \mu + \sum_{j=0}^{\infty} \phi_j \varepsilon_{t-j} \text{ with } \sum_{j=0}^{\infty} |\phi_j| < \infty \text{ by stationarity. Lastly, pages 187-188 in Hamilton demonstrate mean square convergence which as we know, implies convergence in probability. This completes the proof.}
Part II – Empirical Questions

GAUSS

The website for the course contains a GAUSS file labeled “lln.prg” designed to introduce you to GAUSS. It is designed to show you how to generate random variables and to check via a Monte Carlo simulation the LLN and the CLT. We will use this program in a computer lab session to play around with it. You may wait until them or if you feel adventurous, you can play around with it in advance. For the purposes of the problem set all I want for now is that you try to run the program for different values of “lambda,” “inc,” “T,” “M,” and if you are feeling more confident, try simulating the data from a different distribution entirely.

The files “lln1.prg” and “lln1.src” do the same as the file “lln.prg” but in a more elegant manner that I will discuss in the computer session.

EViews

I have created the file “ps1.wf1” which contains a small number of macro time series, mostly with the objective of showing you how EViews works. Feel free to play around with it – for example, you may try graphing the series, obtaining basic statistics, transform into growth rates, etc.