PROBLEM SET 1 – DUE: JANUARY 22, 2004

Instructions

This problem set is divided into two parts: (1) Analytical Questions, and (2) Applied Questions. Part 1, Analytical Questions, should be attempted by each student individually. Part 2, Applied Questions, can be done in collaboration with another partner or alone. I have often found collaboration in the computer room to be very useful. However, if you rely on your partner to do your work you will ensure that you do not learn adequate computer skills nor properly understand the material presented in class.

Please try to answer the questions rigorously by stating any implied assumptions and ensuring all the steps to your conclusion have been properly verified.

Part I – Analytical Questions

1. Suppose
\[ \sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{L} N(0, \sigma^2) \]
Does it follow then that \( \hat{\theta}_T \xrightarrow{p} \theta_0 \)? Explain your answer. Hint:
\[ \hat{\theta}_T - \theta_0 = \frac{1}{\sqrt{T}} \sqrt{T}(\hat{\theta}_T - \theta) \]
and \( p \lim_{T \to \infty} \frac{1}{\sqrt{T}} = 0 \)

2. Combining the delta method and Lindeberg-Levy.
Let \( \{z_t\} \) be an i.i.d. sequence of random variables with \( E(z_t) = \mu \neq 0 \) and \( V(z_t) = \sigma^2 \). Let \( \bar{z}_T \) be the sample mean, derive the asymptotic distribution of \( \frac{1}{\bar{z}_T} \).

3. Show that a random walk is not stationary (hint: check the variance).

4. Suppose \( X_T \xrightarrow{p} c \) and \( Y_T \xrightarrow{p} Y \) where \( Y \sim N(\mu, \sigma^2) \). Derive the limiting distribution of \( X_TY_T \). Be explicit about your assumptions.

5. Does a martingale difference sequence have to be covariance stationary? Explain.

6. Find the orders \( o(.) \) and \( O(.) \) of the following sequences:
• \( b_T = 4 + 2T + 6T^2 \)
• \( b_T = (-1)^T \)
• \( b_T = \exp(-T) \)
• \( b_T = \exp(T) \)

7. Let \( \{x_t\} \) be a sequence of random variables such that \( E(x_t) = \mu \), and \( V(x_t) = \sigma^2 \) for all \( t \), and \( \text{COV}(x_t, x_\tau) = 0 \) for all \( t \neq \tau \). Show that:

\[
\bar{X}_T \xrightarrow{p} \mu
\]

8. Let \( \{u_t\} \) be a sequence of i.i.d. random variables uniformly distributed on \([0, 1]\) and let the random variable \( X \sim N(0, I) \) and independent for all \( t \). Let \( Y_t = X + u_t \) (note \( X \) has no subscript \( t \)). Show that:

- \( \{Y_t\} \) is stationary
- \( \bar{Y}_T \) does not converge in probability to \( \frac{1}{2} \)
- \( \bar{Y}_T - X \xrightarrow{a.s.} 1/2 \)

9. Let \( Y_t = \sum_{j=0}^{\infty} \varphi_j \varepsilon_{t-j} \) where \( \sum_{j=0}^{\infty} |\varphi_j| < \infty \) and \( \{\varepsilon_t\} \) is a martingale difference sequence with \( E(\varepsilon_t^2) = \sigma^2 \). Is \( Y_t \) covariance stationary? Justify your answer.

10. Let \( Y_t \) follow the process

\[
(1 - \phi_1L - ... - \phi_p L^p)(Y_t - \mu) = (1 + \theta_1 L + ... + \theta_q L^q) \varepsilon_t
\]

with the roots of the AR and MA polynomials outside the unit circle (i.e. the process is stationary and invertible). Suppose \( E(\varepsilon_t) = 0 \) for all \( t \), uncorrelated and \( E(\varepsilon_t^2) = \sigma^2 \) and \( E(\varepsilon_t^4) < \infty \). Prove:

\[
\frac{1}{T} \sum_{t=1}^{T} Y_t \xrightarrow{p} \mu
\]
Part II – Empirical Questions

GAUSS

The website for the course contains a GAUSS file labeled “lln.prg” designed to introduce you to GAUSS. It is designed to show you how to generate random variables and to check via a Monte Carlo simulation the LLN and the CLT. For the purposes of the problem set all I want for now is that you try to run the program for different values of “lambda,” “inc,” “T,” “M,” and if you are feeling more confident, try simulating the data from a different distribution entirely.

The files “lln1.prg” and “lln1.src” do the same as the file “lln.prg” but in a more elegant manner.