**PROBLEM SET 2 – DUE: FEBRUARY 3**

**Instructions**

This problem set is divided into two parts: (1) Analytical Questions, and (2) Applied Questions. Part 1, Analytical Questions, should be attempted by each student **individually**. Part 2, Applied Questions, can be done in collaboration with another partner or alone. I have often find collaboration in the computer room to be very useful. However, if you rely on your partner to do your work you will ensure that you do not learn adequate computer skills nor properly understand the material presented in class.

Please try to answer the questions rigorously by stating any implied assumptions and ensuring all the steps to your conclusion have been properly verified.

**Part I – Analytical Questions**

**Problem 1:** Consider the AR(2) process

\[ y_t = 2.5 + 1.1y_{t-1} - 0.18y_{t-2} + \varepsilon_t; \quad \varepsilon_t \sim i.i.d.(0,1) \]

(a) Show that the AR(2) is stable/stationary and calculate its autocovariance and autocorrelation function. Also, calculate the unconditional mean of the process. Indicate what the ACF and the PACF of this series looks like. Do not calculate this by hand. Rather use your favorite econometric software to find the answer for up to 12 lags.

(b) Determine the MA(\infty) representation of this AR(2). This will determine the sequence of dynamic multipliers required for the impulse response function. Display these coefficients graphically (up to 12 lags) using your favorite econometric software.

(c) Determine the forecasts and forecast error variances for the first 12 period-ahead forecasts. Use your favorite econometric software package to answer this question.

**Problem 2:** If

\[ y_t = \mu + x_t + z_t \]
\[ x_t = \varepsilon_t + \theta \varepsilon_{t-1} \]
\[ z_t = \beta \varepsilon_{t-1} + u_t \]

with \( \varepsilon_t \) and \( \eta_t \) independent of each other. Then:

\( \varepsilon_t \sim D(0,\sigma^2) \)
\( \theta \sim D(0,\phi^2) \)

(a) What is the process for \( y_t \)?
(b) Give conditions to ensure $y_t$ is covariance stationary and invertible.

(c) Find the long-horizon forecast for $y_t$ and its variance.

Problem 3: Suppose

$$y_t = \beta y_{t-1} + s_t \varepsilon_t$$

$$s_t = \exp(\beta y_{t-1})$$

$\varepsilon_t \sim N(0, \sigma^2)$

(a) Derive the conditional log-likelihood for $y_0 = 0$.

(b) Derive the score. Assume $\sigma^2$ is known.

(c) Derive the estimator of the information matrix. Assume $\sigma^2$ is known.

(d) Derive the LM test for $H_0: \beta = 0$. Assume $\sigma^2$ is known.

(e) Would it matter if $\sigma^2$ were unknown?

Problem 4: Suppose

$$y_t = f(x_t, \alpha, \beta) + \varepsilon_t$$

with

$$f(x_t, \alpha, \beta) = \beta \ln(\alpha + x_t)$$

Describe an easy approach to testing the hypothesis $H_0: \alpha = 0$ and show how you would conduct the test.

Problem 5: Consider the following stationary data generation process for a random variable $y_t$

$$y_t = \beta y_{t-1} + \varepsilon_t$$

with $\varepsilon_t \sim N(0,1) \text{ i.i.d.}$

with $|\beta| < 1$, and $y_0 \sim N(0, (1 - \beta^2))$.

(a) Obtain the population mean, variance, autocovariances and autocorrelations.

(b) Derive

(i) $E[T^{-1} \sum_{t=2}^{T} y_{t-1} \varepsilon_t] =$

(ii) $E[T^{-1} \sum_{t=1}^{T} y_t^2] =$
(iii) $E\left[ T^{-1} \sum_{t=2}^{T} y_t y_{t-1} \right] = \frac{3 + 6\beta^2 \sigma^2}{1 - \beta^2}.

(iv) Note: $E\left( y_t^4 \right) = \frac{3 + 6\beta^2 \sigma^2}{1 - \beta^4}$. Derive $V\left[ T^{-1} \sum_{t=1}^{T} y_t^2 \right] = \frac{12\beta^2 \sigma^4}{(1 - \beta^2)^2}$.

(c) Derive the limiting distribution of the sample mean. Hint: $\{y_t\}$ is not i.i.d. but it is ergodic for the mean.

(d) Obtain the limiting distribution of the least squares estimator of $\beta$. Hint: calculate $\sqrt{T}(\hat{\beta} - \beta)$ and then derive its distribution.

Problem 6: Consider the AR(1) model of the previous exercise but suppose instead that $e_t \sim N(0, \sigma^2)$. The conditional density for observation $t$ is therefore

$$\log f(y_t | y_{t-1}; \beta, \sigma^2) = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\sigma^2) - \frac{(y_t - \beta y_{t-1})^2}{2\sigma^2}$$

Let $\left( \hat{\beta}, \hat{\sigma}^2 \right)$ be the unrestricted MLE estimate of $\theta = (\beta, \sigma^2)'$ and let $\left( \tilde{\beta}, \tilde{\sigma}^2 \right)$ be the restricted MLE estimate subject to the constraint $R\beta = c$, where $R$ and $c$ are known constants. Also, let

$$\hat{\Sigma} = \begin{bmatrix} \frac{1}{\hat{\sigma}^2} T \sum_{t=2}^{T} y_{t-1}^2 & 0 \\ 0 & \frac{1}{2(\hat{\sigma}^2)^2} \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} \frac{1}{\tilde{\sigma}^2} T \sum_{t=2}^{T} y_{t-1}^2 & 0 \\ 0 & \frac{1}{2(\tilde{\sigma}^2)^2} \end{bmatrix}$$

(a) Verify that $\hat{\beta}$ minimizes the sum of squared residuals so that if it is the same as the OLS estimator.

(b) Verify that $\tilde{\beta}$ minimizes the sum of squared residuals subject to $R\beta = c$ so that it is the restricted LS estimator.

(c) What assumptions did you make on $f(y_t | \beta, \sigma)$ assuming $y_0$ is unobserved? Discuss the importance of any simplifying assumptions and the importance that the parameter $\beta$ has if its value were unrestricted.
(d) Let

\[ Q_T(\theta) = \frac{1}{T} \sum_{t=2}^T \log f(y_t \mid y_{t-1}; \beta, \sigma^2) \]

Show that

\[ Q_T(\hat{\theta}) = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log \frac{SSR_U}{T} \]
\[ Q_T(\tilde{\theta}) = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log \frac{SSR_R}{T} \]

Where \( SSR_U = \sum_{t=2}^T (y_t - \hat{\beta}y_{t-1})^2 \) and \( SSR_R = \sum_{t=2}^T (y_t - \tilde{\beta}y_{t-1})^2 \). Hint: Show that \( \hat{\sigma}^2 = \frac{SSR_U}{T} \) and \( \tilde{\sigma}^2 = \frac{SSR_R}{T} \).

(e) Verify that the \( \hat{\Sigma} \) given above, although not the same as \(-\frac{1}{T} \sum_{t=2}^T H(y_t; \hat{\theta})\), is consistent for \(-E[H(y_t; \theta_0)]\). Verify that \( \tilde{\Sigma} \), although not the same as \(-\frac{1}{T} \sum_{t=2}^T H(y_t; \tilde{\theta})\), is consistent for \(-E[H(y_t; \theta_0)]\). Hint: You may assume that \( \hat{\theta} \) is consistent for \( \theta_0 \) and \( \tilde{\theta} \) is consistent for \( \theta_0 \) under the null. This should make proving consistency of \( \hat{\sigma}^2 \) and \( \tilde{\sigma}^2 \) easier.

(f) Show that the Wald, LM, and LR statistics, using \( \hat{\Sigma} \) and \( \tilde{\Sigma} \) can be written as

\[ W = T \frac{(R\hat{\beta} - c)^2}{(R^2 y_{t-1}^2) SSR_U} \]
\[ LR = T \left\{ \log \left( \frac{SSR_R}{T} \right) - \log \left( \frac{SSR_U}{T} \right) \right\} \]
\[ LM = T \frac{(Y_t' - \hat{\beta}Y_{t-1})Y_{t-1}(Y_t' - \hat{\beta}Y_{t-1})^{-1}Y_{t-1}(Y_t' - \tilde{\beta}Y_{t-1})}{SSR_R} \]
(g) Show that these three statistics can also be written as

\[
W = T \frac{SSR_R - SSR_U}{SSR_U} \\
LR = T \log \left( \frac{SSR_R}{SSR_U} \right) \\
LM = T \frac{SSR_R - SSR_U}{SSR_U}
\]

Problem 7: Consider the stochastic process \{x_t\} which describes the number of trades per interval of time of a particular stock. Thus, \(x_t\) is integer-valued and non-negative. The Poisson distribution is commonly used to describe this type of process. Its density has the form:

\[
f(x_t = j \mid \Omega_{t-1}; \theta) = \frac{e^{-\lambda_t} \lambda_t^j}{j!}
\]

with conditional mean

\[
E(x_t \mid x_{t-1}, \ldots) = \lambda_t = \exp(\alpha + \beta x_{t-1})
\]

Answer the following questions:

(a) Under what parametric restrictions of \(\alpha\) and \(\beta\) will \{x_t\} be stationary?

(b) Write down the log-likelihood function for this problem conditional on \(x_1\).

(c) Compute the first order conditions and obtain the estimators for \(\alpha\) and \(\beta\).

(d) Suppose \(x_t\) is not Poisson distributed. Are \(\alpha\) and \(\beta\) consistently estimated by the conditional log-likelihood in (b)?

(e) Compute the one-step ahead forecast \(x_{t+1} \mid t\). Next, describe the Monte Carlo exercise that would allow you to compute the two-step ahead forecast. Finally, describe what approach would you take if the data were not Poisson distributed but you wanted to produce multi-step ahead forecasts with the conditional mean estimated in (b).

Problem 8: Let \{y_t\} be a stationary zero-mean time series. Define

\[
x_t = y_t - 0.4y_{t-1} \\
w_t = y_t - 2.5y_{t-1}
\]
(a) Express the autocovariance functions for \( \{x_t\} \) and \( \{w_t\} \) in terms of the autocovariance function of \( \{y_t\} \). (Note: there is no assumption that \( \{y_t\} \) is white noise, only covariance stationary).

(b) Show that \( \{x_t\} \) and \( \{w_t\} \) have the same autocorrelation functions.

(c) Show that the process \( u_t = -\sum_{j=1}^{\infty} 0.4^j x_{t+j} \) satisfies the difference equation
\[
u_t - 2.5u_{t-1} = x_t
\]

Problem 9: Find the largest values for \( |\rho_1| \) and \( |\rho_2| \) for an MA(2) model.

Problem 10: Consider an AR(1) process, such as
\[y_t = c + \phi y_{t-1} + \varepsilon_t\]
where the process started at \( t = 0 \) with the initial value \( y_0 \). Suppose \( |\phi| < 1 \) and that \( y_0 \) is uncorrelated with the subsequent white noise process \( (\varepsilon_1, \varepsilon_2, \ldots) \).

(a) Write the variance of \( y_t \) as a function of \( \sigma^2 (= V(\varepsilon)) \); \( V(y_0) \), and \( \phi \). Hint: by successive substitution show
\[y_t = (1 + \phi + \phi^2 + \ldots + \phi^j)c + \varepsilon_t + \phi \varepsilon_{t-1} + \ldots + \phi^j \varepsilon_1 + \phi^j y_0\]

(b) Let \( \mu \) and \( \{\gamma_j\} \) be the mean and the autocovariances of the covariance stationary AR(1) process, so that
\[\mu = \frac{c}{1-\phi} \quad \gamma_j = \frac{\phi^j \sigma^2}{1-\phi^2}\]

Show that:
(i) \( \lim_{t \to \infty} E(y_t) = \mu \)

(ii) \( \lim_{t \to \infty} V(y_t) = \gamma_0 \)

(iii) \( \lim_{t \to \infty} COV(y_t, y_{t-j}) = \gamma_j \)
Part II – Applied Questions

This part of the problem set will involve manipulating an EViews file called “ma1.wf1” and two GAUSS files called “ma1.prg” and “newma1.prg” respectively. All files can be downloaded from my web site. The file “ma1.wf1” contains a LogL object. This object allows you to specify the likelihood for a model and maximize this likelihood, obtain parameter estimates and standard errors. I have provided a simulated MA(1) model with $\mu = 2$, $\theta = 0.5$, $\sigma = 1$. This is the series $y$. The series $e$ contains the error terms that I used to construct the series $y$. The LogL object specifies the conditional likelihood of an MA(1). You should check the estimates that you get from this procedure and those you get from using the built-in routines. They should be the same.

The file ma1.prg is a GAUSS program that estimates the conditional likelihood of an MA(1) model. It uses the add-on MAXLIK, which is already installed in the UNIX machines. This is a bare-bones program but it should be trivial to estimate the series $y$ generated with EViews and check that you get the same estimates. The file newma1.prg is a GAUSS program that estimates the same conditional likelihood as the file ma1.prg but uses the add-on OPTMUM instead. This is a general optimizer so it does not automatically provide statistics associated with inference. Consequently, this file contains an additional section dedicated to computing the statistics. I encourage you to use both files and check that you get the same results.

The goal of this part of the exercise is to manipulate the above programs to construct your own estimators of ARMA models. In particular:

(a) Simulate an ARMA(0,2) and an ARMA(1,1) using either EViews or GAUSS. Use the parameter values of your choosing. Disregard the first 100 observations and create series of length 300.

(b) Modify the EViews and GAUSS programs appropriately (try to use “procs” in a master and library files!) and estimate these models by MLE (you may use the conditional MLE if you wish. Brownie points will be awarded if you estimate the models with the exact likelihood). Experiment with different algorithms, starting values, etc.

(c) Report the results of your estimation exercise. At a minimum you should comment on:

- Speed of convergence of the different algorithms used. This requires that you modify options in EViews and GAUSS.
- Estimation problems encountered and how did you solve them.
- Robustness of your estimates -- did the estimation procedures converge to the same point using different initial values?
- Estimation Output.
• Printouts of the modified GAUSS programs and of the modified LogL objects in EViews.

My web site also contains information on the LogL object in EViews and the MAXLIK library in GAUSS.