MIDTERM – FEBRUARY 19, 2004

Instructions

You should attempt all questions in this exam. If you are unable to give analytical solutions to the problems, credit will be given if you manage to suggest an adequate numerical solution technique or at least some intuition about the expected result. Be specific in the assumptions and steps that you take. This will earn you partial credit if your answer is incorrect further down the line. However, you need not prove every single result as long as it is a well established result or it is in your class notes. I will demand a high standard of precision and clarity. Sloppy presentation will almost surely result in a poor score. Please invest time in presenting your work in an intelligible and neat manner. Good luck!

Please answer in the space provided for each question as neatly as possible. To ensure neatness, please first work out the solution in scratch paper and then enter the answer in the appropriate space. The exam is for a total score of 100 points allocated as indicated for each question.

Problem 1 – Shorter Questions

Find the orders $o(.)$, $O(.)$, $o_p(.)$, and $O_p(.)$ for the following sequences:

(i) $b_T = \sum_{t=1}^{T} t^\delta$ Solution: $o(T^{\delta+1})$, $O(T^{\delta+1})$, $\delta > 1$

(ii) $b_T = \sum_{t=1}^{T} \alpha^t$ $|\alpha| < 1$. Solution: $o(T^\delta)$, $O(1)$, $\delta > 0$

(iii) $b_T = \sum_{t=1}^{T} u_t \overset{iid}{\sim} D(0, \sigma^2)$ Solution: $o_p(T^{\delta/2})$, $O_p(T^{1/2})$, $\delta > 1$

(iv) $b_T = \sum_{t=1}^{T} tu_t \overset{iid}{\sim} D(0, \sigma^2)$ Solution: $o_p(T^{3\delta/2})$, $O_p(T^{3/2})$, $\delta > 1$

(v) $b_T = \sum_{t=1}^{T} u_t^2 \overset{iid}{\sim} D(0, \sigma^2)$ Solution: $o_p(T^{5\delta/2})$, $O_p(T^{1/2})$, $\delta > 1$

Problem 2

Consider the model

$y_t = \delta + \varepsilon_t$

$\varepsilon_t = \rho \varepsilon_{t-1} + u_t \overset{iid}{\sim} D(0, \sigma^2) \ | \rho | < 1; u_t \sim D(0, \sigma^2)$

(a) Derive the asymptotic distribution of the OLS estimator for $\delta$. Make sure to be explicit about any assumptions you make. Hints:

$$\frac{1}{T^{v+1}} \sum_{t=1}^{T} t^v = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{t}{T} \right)^v \longrightarrow \frac{1}{v+1}$$
Solution:

\[
\hat{\delta} = \frac{\sum_{t=1}^{T} t y_t}{\sum_{t=1}^{T} t^2} = \frac{\sum_{t=1}^{T} t(\delta t + \varepsilon_t)}{\sum_{t=1}^{T} t^2} = \delta + \frac{\sum_{t=1}^{T} t \varepsilon_t}{\sum_{t=1}^{T} t^2}
\]

using the hint notice that,

\[
\frac{1}{T^3} \sum_{t=1}^{T} t^2 \longrightarrow \frac{1}{3}
\]

This solves the convergence problem for the denominator. Now consider the numerator,

\[
\sum_{t=1}^{T} \frac{t}{T} \varepsilon_t = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{1 - \rho L} \right) u_t
\]

Using the Beveridge-Nelson decomposition trick to the previous expression,

\[
\sum_{t=1}^{T} \frac{t}{T} \left( \frac{1}{1 - \rho L} \right) u_t = \left( \frac{1}{1 - \rho} \right) \sum_{t=1}^{T} \frac{t}{T} u_t + \psi^*(L) u_T
\]

where the last term is left implicit since it will vanish in probability by the usual arguments. Hence, since

\[
\sum_{t=1}^{T} \frac{t}{T} u_t \overset{d}{\longrightarrow} N \left( 0, \frac{\sigma^2}{3} \right)
\]

\[
\psi^*(L) U_T \overset{p}{\longrightarrow} 0
\]

then

\[
\sum_{t=1}^{T} \frac{t}{T} \varepsilon_t \overset{d}{\longrightarrow} N \left( 0, \frac{\sigma^2 (1 - \rho)^2}{3(1 - \rho)^2} \right)
\]

Putting the results for the numerator and the denominator together, we obtain the desired distribution,

\[
T^{3/2} \left( \hat{\delta} - \delta \right) \overset{d}{\longrightarrow} N \left( 0, \frac{3\sigma^2}{(1 - \rho)^2} \right)
\]

QED
(b) Explain why this estimator is superconsistent. Discuss the implication of this property.

Solution: The estimator for $\delta$ converges at rate $T^{3/2}$ instead of the usual $T^{1/2}$ rate. Hence, the estimator is converging at a faster rate to the true parameter value. The implications of this result are that $\delta$ is therefore more precisely estimated (since its variance is also converging more rapidly) and is therefore less susceptible to omitted terms in the specification of the dynamic structure of the residuals.

(c) Given that the estimator is superconsistent, what is the distribution of the usual t-ratio? You may use heuristic arguments to answer this question somewhat less formally.

Solution:

The t-ratio is:

$$t = \frac{\hat{\delta} - \delta}{\hat{\sigma}_\delta}$$

From part (a) we know that,

$$T^{3/2}(\hat{\delta} - \delta) \xrightarrow{d} N\left(0, \frac{3\sigma^2}{(1 - \rho)^2}\right)$$

We need to show that the denominator of the t-ratio is consistent for the true population moment.

Notice that,

$$\hat{\sigma}_u = \hat{\sigma}_\varepsilon \left(\sum_{t=1}^T t^2\right)^{-1}$$

with

$$\hat{\sigma}_\varepsilon \xrightarrow{p} \frac{\sigma_u^2}{(1 - \rho)^2}$$

$$\frac{1}{T^3} \sum_{t=1}^T t^2 \xrightarrow{p} \frac{1}{3}$$

hence, the ratio is converging at the usual $T^{1/2}$ rate to a $N(0,1)$ which is the usual asymptotic result for the t-ratio.
Problem 3:
Consider the following process,

\[ y_t = \ln(\beta y_{t-1}) + \varepsilon_t \quad y_0 = 1; \quad \varepsilon_t \sim iid \quad \varepsilon_t \sim D(0,1), \quad \beta > 0 \]

(a) Discuss how you would determine whether this process is stationary (although it is not necessary to derive this analytically, it is a bonus if you can determine for what values of \( \beta \) will the process be stationary).

Solution:

One approach is to try to propagate the process forward for different values of \( \beta \). It turns out that for this Mickey Mouse example, the process is stationary for any \( \beta \).

(b) Assuming the regularity conditions for QMLE are met, find the conditional QMLE estimator for \( \beta \) (i.e., conditional on \( y_0 = 1 \)). Assume the variance of the \( \varepsilon \) is known to be 1.

Solution:

Assuming normality,

\[ Q(\beta) = -\frac{T}{2} \log(2\pi) - \frac{T}{2} \log(1) - \frac{1}{2} \sum_{t=1}^{T} \left( y_t - \ln(\beta y_{t-1}) \right)^2 \]

First order conditions,

\[ \frac{\partial Q}{\partial \beta} = \frac{1}{\beta} \sum_{t=1}^{T} \left( y_t - \ln(\beta y_{t-1}) \right) \]

\[ \hat{\beta} = \exp\left\{ \bar{y} - \ln(\bar{y}) \right\} \]

(c) Assuming the regularity conditions for QMLE are met, derive the asymptotic distribution for \( \hat{\beta} \). Note, you do not need to derive this from first principles, you may use QMLE results.

Solution:

\[ \frac{\partial^2 Q}{\partial \beta^2} = -\frac{T}{\beta^2} \frac{1}{\beta^2} \sum_{t=1}^{T} \left( y_t - \ln(\beta y_{t-1}) \right) \]

\[ \frac{\partial^2 Q}{\partial \beta^2} = -\frac{T}{\beta^2} \frac{1}{\beta^2} \sum_{t=1}^{T} \left( y_t - \ln(\beta y_{t-1}) \right) \]
By QMLE results and assuming the regularity conditions are met, we know that

\[ p \lim \left( - \frac{1}{T} \frac{\partial^2 Q}{\partial \beta^2} \right) = \frac{1}{\beta^2}; \]

\[ p \lim \left( \frac{1}{T} \frac{\partial Q}{\partial \beta} \frac{\partial Q}{\partial \beta'} \right) = \frac{1}{\beta^2} \quad \text{since } \sigma^2 = 1 \]

and therefore

\[ \sqrt{T} (\hat{\beta} - \beta) \xrightarrow{d} N(0; \beta^2) \]

(d) Given the distribution of \( \hat{\beta} \), find the distribution of \( \ln \hat{\beta} \).

**Solution:** Notice that,

\[ \lambda(\beta) = \ln \beta \]

\[ \lambda'(\beta) = \frac{1}{\beta} \]

Using the delta method,

\[ \sqrt{T} (\ln \hat{\beta} - \ln \beta) \xrightarrow{d} N\left(0; \frac{1}{\beta} \frac{1}{\beta} \right) = N(0, 1) \]

Notice that this is the result is equivalent to the result we would obtain from the least squares regression

\[ y_t - \ln y_{t-1} = \gamma + \varepsilon_t \]

with \( \gamma = \ln \hat{\beta} \) and since by assumption \( \sigma^2 = 1 \), the result immediately follows.

(e) Instead, given \( \hat{\beta} \) from QMLE, describe in steps how you would construct a 95% confidence interval for \( \ln \hat{\beta} \) with the bootstrap.

**Solution:**

Step 1. Draw with replacement a sample of size \( T \) from the residuals \( \hat{\varepsilon}_t \), re-centered.

Step 2. Given \( \hat{\beta} \) and the initial condition, reconstruct the bootstrap sample of size \( T \) of \( y^*_t \).

Step 3. Estimate the model and obtain \( \hat{\beta}^b \).
Step 4. Replicate steps 1-3, B times.

The question does not ask that you obtain an asymptotic refinement so you can construct the confidence interval directly from the B estimates $\hat{\beta}^b$ and choose the 5th and 95th percentiles. Alternatively, even though the variance of $\hat{\beta}$ is a function of $\beta$, the t-ratio is a pivotal statistic and hence a 95% confidence interval can be constructed with the percentile t-method by obtaining B estimates of the t-ratio, say $t^*$ and constructing the 95% confidence interval as

$$[\hat{\beta} - t^*[0.025]S_{\hat{\beta}}; \hat{\beta} - t^*[0.025]S_{\hat{\beta}}]$$

(f) Suppose instead that you are asked to construct a 95% confidence interval with the bootstrap for $\ln \beta$ from the least squares regression,

$$y_i - \ln y_{i-1} = \ln \beta + \epsilon_i$$

with $\sigma_\epsilon = 1$ and known. Is the percentile method pivotal? Does the bootstrap provide an asymptotic refinement in this case?

**Solution:**

In this case, notice that

$$\sqrt{T}(\ln \hat{\beta} - \ln \beta) \xrightarrow{d} N(0,1)$$

so that the percentile method is pivotal (since the variance is known and equal to one) and the bootstrap provides an asymptotic refinement.

(g) For an initial guess for $\beta$, say $\beta_1$, construct the first step in the BHHH algorithm.

**Solution:**

BHHH step

$$\hat{\beta}_{j+1} = \hat{\beta}_j + \left(1 - \frac{\partial Q}{\partial \beta} \frac{\partial Q}{\partial \beta'} \right)^{-1} \frac{\partial Q}{\partial \beta} \left|_{\hat{\beta}_j} \right.$$  

$$\hat{\beta}_2 = \hat{\beta}_1 + \left( \frac{\sum_{i=1}^T (y_i - \ln(\hat{\beta}_j y_{i-1}))^2}{T \hat{\beta}_1^2} \right) \left( \frac{\sum_{i=1}^T (y_i - \ln(\hat{\beta}_j y_{i-1}))}{\hat{\beta}_1} \right)$$