Problem Set 5 – Due: March 11

Instructions

This problem set is divided into two parts: (1) Analytical Questions, and (2) Applied Questions.

Please try to answer the questions rigorously by stating any implied assumptions and ensuring all the steps to your conclusion have been properly verified.

Part I – Analytical Questions

Problem 1: Consider the following DGP

\[
\begin{align*}
x_t + \beta y_t &= u_{1t}; & \text{where } u_{1t} &= \theta u_{1t-1} + \varepsilon_{1t} \\
x_t + \alpha y_t &= u_{2t}; & \text{where } u_{2t} &= \rho u_{2t-1} + \varepsilon_{2t}
\end{align*}
\]

with \(|\rho| < 1\) and

\[
\begin{pmatrix}
\varepsilon_{1t} \\
\varepsilon_{2t}
\end{pmatrix}
\sim D\left[
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\begin{pmatrix}
\sigma_1^2 & \gamma \\
\gamma & \sigma_2^2
\end{pmatrix}
\right]
\]

where \(D\) denotes a generic distribution.

(a) Derive the degree of integratedness of the two series, \(x_t\) and \(y_t\), explicitly stating the parameter restrictions required in each case.

(b) Under what coefficient restrictions are \(x\) and \(y\) cointegrated? What are the cointegrating vectors in such cases?

(c) Choose a particular set of coefficients that ensures \(x\) and \(y\) are cointegrated and derive the following representations:

(i) the moving-average: that is, \((x_t, y_t)'\) on the left hand side, \((\varepsilon_{1t}, \varepsilon_{2t})'\) and its lags on the right hand side.

(ii) The autoregressive in the levels: that is, \((x_t, y_t)'\) on the right hand side, and lags of \((x_t, y_t)'\) and \((\varepsilon_{1t}, \varepsilon_{2t})'\) on the right hand side.

(iii) The Error-Correction Representation: that is, \((\Delta x_t, \Delta y_t)'\) as a function of \(z_{t-1}\) and residuals (no need to be specific about the residuals).
(d) [5 points] In general, discuss the pros and cons of obtaining impulse responses from a VAR estimated in the levels, as in part (c)-(i) and one in vector error correction form, as in part (c)-(iii).

Problem 2: Consider the following D.G.P.

\[
x_t + y_t = v_t, \quad v_t (1 - \rho_1 L) = \varepsilon_{1t} \\
2x_t + y_t = u_t, \quad u_t (1 - \rho_2 L) = \varepsilon_{2t}
\]

and

\[
\begin{pmatrix}
\varepsilon_{1t} \\
\varepsilon_{2t}
\end{pmatrix}
\sim N
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\begin{pmatrix}
\sigma_1^2 & 0 \\
0 & \sigma_2^2
\end{pmatrix}
\]

(a) Determine whether this system is stationary, non-stationary, or non-stationary but cointegrated according to the following scenarios:

- (i) $|\rho_1| < 1, |\rho_2| < 1$
- (ii) $\rho_1 = 1, |\rho_2| < 1$
- (iii) $\rho_1 = 1, \rho_2 = 1$

(b) Obtain the reduced-form, autoregressive representation of the system in the levels when it is cointegrated.

(c) Given the representation that you just found, calculate the reduced-form, impulse response function coefficient matrices, $\psi_s$ for periods $s = 0, 1, \text{and } 2$ for $\rho_2 = 0.5$. What is $\psi_s$ as $s \to \infty$? Given the reduced form impulse response matrices, calculate the structural impulse responses for periods 0, 1, 2. What is happening as $s \to \infty$? Explain this result.

(d) Find the moving-average representation of the system and the cointegrating vector when it is cointegrated.

(e) Describe how you could estimate the cointegrating vector in a regression of $y_t$ on $x_t$ which is well behaved in small samples.

Problem 3: Consider the following bivariate VAR

\[
\begin{align*}
y_{1t} &= 0.3y_{1t-1} + 0.8y_{2t-1} + \varepsilon_{1t} \\
y_{2t} &= 0.9y_{1t-1} + 0.4y_{2t-1} + \varepsilon_{2t}
\end{align*}
\]
with $E(e_t,e_{1t}) = 1$ for $t = \tau$ and 0 other wise, $E(e_{2t},e_{2t}) = 2$ for $t = \tau$ and 0 other wise, and $E(e_t,e_{2t}) = 0$ for all $t$, and $\tau$. Answer the following questions:

(a) Is this system covariance-stationary?

(b) Calculate $\psi_s = \frac{\partial y_t}{\partial \epsilon_{1t}}$ for $s = 0, I, \text{and} 2$. What is the limit as $s \to \infty$?

(c) Calculate the fraction of the MSE of the two period-ahead forecast error for variable 1, $E\left[y_{1,t+2} - \hat{E}(y_{1,t+2} | y_t, y_{t-1}, ...)^2\right]$, that is due to $\epsilon_{1,t+1}$ and $\epsilon_{1,t+2}$.

Problem 4: Consider the following VAR

$$y_t = (1 + \beta)y_{t-1} - \beta\alpha x_{t-1} + \epsilon_{1t}$$
$$x_t = \beta y_{t-1} + (1 - \gamma\alpha)x_{t-1} + \epsilon_{2t}$$

(a) Show that this VAR is not-stationary.

(b) Find the cointegrating vector and derive the VECM representation.

(c) Transform the model so that it involves the error correction term (call it $z$) and a difference stationary variable (call it $\Delta w_t$). $w$ will be a linear combination of $x$ and $y$ but should not contain $z$. Hint: the weights in this linear combination will be related the coefficients of the error correction terms.

(d) Verify that $y$ and $x$ can be expressed as a linear combination of $w$ and $z$. Give an interpretation as a decomposition of the vector $(y'x)'$ into permanent and transitory components.

Problem 5: Consider the bivariate VECM

$$\Delta y_t = c + \alpha \beta' y_{t-1} + \epsilon_t \sim iid (0, \sigma^2)$$

where $\alpha = (\alpha_1,0)'$ and $\beta = (1,-\beta_2)'$. Equation by equation, the system is given by

$$\Delta y_{1t} = c_1 + \alpha_1 (y_{1t-1} - \beta_2 y_{2t-1}) + \epsilon_{1t}$$
$$\Delta y_{2t} = c_2 + \epsilon_{2t}$$

Answer the following questions:

(a) From the VECM representation above, derive the VECM representation

$$\Delta y_t = c + \Pi y_{t-1} + \epsilon_t$$
and the VAR(1) representation

\[ y_t = c + Ay_{t-1} + \varepsilon_t \]

(b) Based on the given values of the elements in \( \alpha \) and \( \beta \), determine \( \alpha_\perp, \beta_\perp \), such that \( \alpha' \alpha_\perp = 0 \) and \( \beta' \beta_\perp = 0 \).

(c) Using the Granger representation theorem determine that

\[ \psi(1) = \beta_\perp (\alpha_\perp' I_2 \beta_\perp')^{-1} \alpha_\perp \]

where \( \psi(L) \) is the moving average polynomial corresponding to the VECM system above and \( I_2 \) is the identity matrix of order 2. *Hint*: you may show this result by showing that \( \psi(1) \) is orthogonal to the cointegrating space.

(d) Using the Beveridge-Nelson decomposition and the result in (c), determine the common trend in the VECM system.

(e) Show that \( \beta' y_t \) follows an AR(1) process and show that this AR(1) is stable provided that \(-2 < \alpha < 0\). What can you say about the system when \( \alpha = 0 \)?

**Part II – Applied Questions**

Problem 1: Use EViews to do this assignment. Gather data from the St. Louis Fed (http://research.stlouisfed.org/fred2/) on the following quarterly variables: nominal GDP (Y), the GDP deflator (P), M2 (M), and the three month T-Bill rate (i). Take the logs of Y, P, and M respectively and call them \( y, p, m \).

(a) Test for cointegration. In particular, (i) test the quantity theory of money relationship with a two-step Engle-Granger cointegration test. Then use a Johansen test to determine the cointegration space. Report your results using tables that specify all the elements of how you conducted the tests.

(b) Do a Granger causality test to determine whether money Granger-causes output. First do the test by omitting \( i \) and its lags and then repeat with \( i \) and its lags. Report your results in a table making sure to describe all the elements of how you conducted the test. Comment on the differences in your results and on the validity of the distribution of your test in lieu of your results in par (a).

(c) Assuming a Wold causal ordering \( y, m, p, i \) calculate the *structural* response of the variables in your system to a structural shock in \( i \). Plot these using a graph for each impulse response but report all 4 graphs in one page. Your graphs should include standard error bands. Repeat the exercise for a structural shock in \( m \) instead (4 graphs, one page, S.E. bands included). Comment on the economic significance of your results.
(d) Estimate the responses of $y$, to a structural shock in $i$ by local projections. Plot in a graph along with S.E. bands and the impulse response from (c). Comment on the differences/similarities.