PROBLEM SET 3 – DUE: JANUARY 31, 2006

Instructions

Please try to answer the questions rigorously by stating any implied assumptions and ensuring all the steps to your conclusion have been properly verified.

Part I – Analytical Questions

1. Suppose $\hat{\pi}_T \rightarrow \pi_0$ and let $g(\hat{\pi}_T; \theta) = \hat{\pi}_T - h(\theta)$, where $h$ is a continuous function and where $V(\hat{\pi}_T) = \sigma^2$.
   (a) Show that $\sqrt{T}(\hat{\theta}_T - \theta_0) \rightarrow^d N(0, (G'WG)^{-1} G'W\sigma^2WG(G'WG)^{-1})$
   where $G = \nabla_\theta g(\hat{\pi}_T; \theta)$ and find the specific version of this expression that applies to this problem with optimal weighting matrix $W = (\sigma^2)^{-1}$.

   (b) Suppose $h(\theta) = 3\theta^2$. Derive the asymptotic distribution for this particular case.

   (c) Instead, use the delta method to derive the asymptotic distribution of $\hat{\theta}_T$ in (b). Verify that you get the same results with both methods.

2. Suppose $y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$. Define $Y = (y_1, ..., y_T)'$ and $M = (\mu, ..., \mu)'$.
   (a) Obtain $\Omega = E(Y - M)(Y - M)'$
   (b) Hence derive the log-likelihood function for $Y$.
   (c) Show that the log-likelihood for $Y$ is equivalent to the log-likelihood based on
   $$\frac{1}{T} \sum_{t=1}^{T} \ln f(y_t; \theta)$$

3. The Hannan-Rissanen estimator for an MA(1) is a two step estimator in which the first step, a long autoregression is fitted to the data and in the second step, the residuals of the first step are used as regressors in the MA(1) specification. Suppose the D.G.P. is
   $$y_t = \varepsilon_t + \theta \varepsilon_{t-1}, \ |\theta| < 1$$
   (a) The first step in the Hannan-Rissanen method consists of estimating
   $$y_t = \rho_1 y_{t-1} + ... + \rho_p y_{t-p} + u_t$$
where \( p \) is fixed (for example, chosen by information criteria). Assuming \( \hat{\rho}_j \to \theta^j \) for \( j = 1, \ldots, p \), derive the expression for \( \hat{u}_t \) and show under what conditions \( \hat{u}_t \to \varepsilon_t \).

(b) The second step consists in estimating \( \theta \) with the regression

\[
y_t = \nu_t + \theta \hat{u}_t
\]

Show that this estimator is consistent for \( \hat{\theta} \).

(c) Derive the asymptotic distribution of \( \hat{\theta} \). Show that this estimator is less efficient than the maximum likelihood estimator.

4. Consider the following ARMA(2,1) model

\[
y_t = 2 + 1.2 y_{t-1} - 0.5 y_{t-2} + \varepsilon_t + 0.25 \varepsilon_{t-1}
\]

(a) Write this model in state-space form

(b) Using this form, verify whether the model is covariance stationary (you may use GAUSS).

(c) Compute the expression for \( E(y_{t+2} \mid y_{t-1}, \ldots) \)

(d) Derive the infinite MA representation for this model.

**COMPUTER PROBLEM**

Coming soon under separate cover.