Handout #1: Review of Economic Terminology and Math for Economists

1. The components of economic models

Economics uses formal models in order to describe and analyze complicated relationships. Useful economic models are ones in which the judicious choice of assumptions highlights the critical interactions between the components of a model. Since we will be exploring models throughout the course, it is important to review some of the terminology associated with models. Two of the most important terms of economic models are: endogenous and exogenous. The first term refers to variables that are determined within the model while the second term refers to variables that are determined outside the model. As an example, in the theory of individual's demand functions, the price of the good, the prices of substitute and complement goods, and agent's income are all treated as exogenous. The only endogenous variable is the quantity demanded.

Another important characterization of models is the type of analysis, specifically whether the analysis is partial equilibrium or general equilibrium. In general equilibrium analysis, the prices and quantities of all goods (if there are multiple markets) are determined — i.e. both prices and quantities are determined endogenously. The exogenous elements are the specification of agent's utility functions and initial endowments and the technology faced by producers. In a partial equilibrium analysis, the prices of all goods are usually treated as exogenous. As an example, in deriving agents' demand functions (represented by a downward sloping demand curve) and producers' supply functions (represented by an upward sloping supply function), prices are taken as given. Note that in partial equilibrium analysis, prices are exogenous. This illustrates an important point: whether a variable is treated as exogenous or endogenous depends on the type of analysis. The reason that the distinction between exogenous and endogenous is important is that these definitions frame the kinds of questions permissible in analyzing the behavior of a model. Namely, we want to see how equilibrium characteristics — defined by the endogenous variables — change when some of the exogenous variables change. As an example, it makes sense to ask what happens to the quantity supplied of a good if the price of that good changes since the price is exogenous in the model of firm supply. However, this is not a well posed question when analyzing a market since market analysis involves general equilibrium analysis and price is endogenous.

Finally, it is useful to remember, when doing graphical analysis, the distinction between movements along a curve and movements of a curve. This difference is due to the fact that most relationships in economics involve more than two dimensions while graphical analysis is, for the most part, restricted to only two dimensions. Hence, a change in one of the variables represented on an axis is a movement along a curve; the location of the curve is determined by the critical variables, which are not represented explicitly on the axes. As an example, the quantity demanded of a good is given by the demand function:

\[ q_i^d = q_i(p_i, p_s, p_y) \]

where \( q_i \) is the quantity of good \( i \), \( p_i \) is the price of good \( i \), \( p_s \) is the price of a complement, \( p_y \) is the price of a substitute and \( y \) is the agent's income. We typically represent this graphically with \( p_i \) and \( q_i \) on the axes; the location of the demand curve is determined by the other three variables.
2. Math for Economists

A. Differentiation
Economic analysis was transformed in the middle of the 19th century by the marginalist revolution. This emphasized that the critical notion in describing economic activity was the change in utility, profits, etc. due to an incremental change in behavior. Hence the importance accorded to concepts like marginal product and marginal cost. It also explains why economists like to use differential calculus — since the definition of a derivative is precisely related to the change in a function due to a change in a variable. Recall the definition of the derivative. Let \( y = f(x) \), then the derivative of \( y \) with respect to \( x \) is defined by:

\[
\frac{dy}{dx} = \frac{df}{dx} = f'(x) = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta}.
\]

Note that the fraction is the “rise over the run” — the derivative is the slope of the function at the point \( x \).

As stated in the discussion above, the relationships studied in economics typically involve more than two dimensions. This requires an extension of the derivative to functions of more than one variable. This extension is the partial derivative in which the above limit involves the change in only one of the variables. That is, let \( y \) be a function of \( n \) variables: \( y = f(x_1, x_2, \ldots, x_n) \). Then, the partial derivative of \( y \) with respect to \( x_i \) is:

\[
\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_i(x_1, x_2, \ldots, x_n) = \lim_{\delta_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \delta_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\delta_i}.
\]

Note that all variables other than \( x_i \) are held constant. This makes partial differentiation so straightforward; in taking the partial derivative, all variables other than the variable of differentiation are treated as constants.

As an example, consider the following production function which we will study more closely in our discussion of growth theory.

Let \( y \) denote output, \( k \) is capital, and \( h \) denotes labor. The relationship between these variables is represented by the following functional form.

\[
y = f(k, h) = k^a h^{1-a}.
\]

Then, the partial derivative of output with respect to capital (the marginal product of capital is): \( \frac{\partial y}{\partial k} = ak^{a-1}h^{1-a} \).

Where derivatives are commonly used is in finding the necessary conditions associated with the maximum of a function; i.e. the first-order conditions. Recall that for a function of one variable, the necessary condition for a maximum is \( f'(x) = 0 \). For most economic environments, the objective function involves two types of entities: those that make people happy (utility or revenue) and those that represent costs (i.e. prices). Hence, the necessary condition associated with maximizing an economic objective function always has the interpretation that \( MC = MB \) (marginal cost \( = \) marginal benefit). We will see this repeatedly.

A final concept from calculus that we will use on occasion is that of a composite function. This is a function in which the argument (or one of the arguments if the original function has more than one argument) is a function itself. That is, let \( y = f(x) \) and \( z = g(y) \). This implies the composite function \( h(x) = g(f(x)) \). The derivative of \( h \) with respect to \( x \) is, therefore determined by two effects: how much \( y \) changes when \( x \) changes \( \left( \frac{df}{dx} \right) \) and how much \( z \) changes when \( y \) changes \( \left( \frac{dg}{dy} \right) \). Hence, the derivative of \( z \) with respect to \( x \) is:

\[
\frac{dz}{dx} = \left( \frac{dg}{dy} \right) \left( \frac{dy}{dx} \right).
\]

This is referred to as the chain rule.

As an example of all this, consider a hypothetical market for oranges. Assume that
the supply of oranges is entirely determined by the quantity of rain, denoted \( \rho \). Rain is exogenous and, hence, the quantity of oranges is exogenous as well. The relationship between rainfall and the quantity of oranges supplied is given by the following function:

\[
q^s = q^s(\rho) = \rho^\alpha; \quad 0 < \rho < 1.
\]

(The plus sign in parentheses denotes that rainfall and oranges are positively related.) The demand for oranges is positively related to agents' income, denoted \( y \), and negatively related to the price of oranges, \( p \). This is expressed as the demand function:

\[
q^d = q^d(y, p) = \frac{y}{p}.
\]

Equilibrium in this market requires that

\[
q^s = q^d.
\]

Before going through the math, let's review the model. Since rain is exogenous and the supply of oranges depends only on rain, then the quantity of oranges on the market in equilibrium is already determined. The location of the demand curve is determined by the level of income which is also exogenous. Assume that income is currently at the level denoted \( y_0 \). Then the equilibrium is where the demand and supply curves intersect — this intersection determines the equilibrium price of oranges. Graphically, the market is depicted as:

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Figure 1: Market for Oranges

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This graph implies that when the quantity of rain is \( \rho_0 \) then, given the location of the demand curve, the equilibrium price will be \( p_0 \). Suppose that the quantity of rainfall is less, say \( \rho_1 \). This would imply that the quantity of oranges supplied would shift to the left and the equilibrium price of oranges would rise so that \( p_1 > p_0 \). (You can supply the graph.) We now want to examine this mathematically. That is, we want to determine the derivative \( \frac{dp}{d\rho} \) (which we already know is negative).

First note that equilibrium price is a composite function of rain: rain determines output, then, through equilibrium and the demand function, equilibrium price is determined. That is:

\[
\frac{dp}{d\rho} = \left( \frac{dp}{dq} \right) \left( \frac{dq}{d\rho} \right)
\]

(Note that I am no longer denoting whether quantity is demanded or supplied since in equilibrium these must be equal.)

The first derivative is given by the demand function (eq. (A.2)) above. Rewrite price as a function of quantity and take the derivative. This yields

\[
\frac{dp}{dq} = -\frac{y}{q^2}.
\]

From the supply relationship (eq. (A.1)), we have

\[
\frac{dq}{d\rho} = \alpha \rho^{\alpha-1}.
\]

Then, using the chain rule results in:

\[
\frac{dp}{d\rho} = -\left( \frac{y}{q^2} \right) (\alpha \rho^{\alpha-1}) = \left( \frac{y}{\rho^{2\alpha}} \right) (\alpha \rho^{\alpha-1}) = -\alpha y \rho^{(1+\alpha)}
\]

The second equality comes from using eq. (A.1) to replace \( q \). Note that this expression is indeed negative.
This is an illustration of the chain rule – but a clumsy one. If we simply wanted to find the derivative of price with respect to rain, the most direct method would be to substitute for $q$ immediately. You should prove to yourself that this does indeed yield eq. (A.7). The chain rule is most helpful when explicit functional forms are not assumed.

One of the ways we will characterize equilibrium phenomena is in terms of percentage changes. Percentage changes are useful since they are unit free – scaling does not matter. In market analysis, a common type of percentage change is that of elasticity. To use this market example a bit more for illustrative purposes, we can calculate the elasticity of equilibrium price with respect to rainfall. Let $\eta$ denote this elasticity; it is defined as:

$$\eta = \frac{\% \Delta p}{\% \Delta \rho} = \frac{(\Delta p/p)}{(\Delta \rho/\rho)} = \frac{\Delta p}{\Delta \rho} \left( \frac{\rho}{p} \right).$$

Interpreting the delta terms as a derivative and using the results from before yields

$$\eta = -\alpha y p^{-(1+\alpha)} \rho p^{-1}.$$

(A.8)

Does this make sense? Again, think of the two effects described by the functional forms for demand and supply. From the demand relationship, price and quantity move inversely 1 for 1 in percentage terms – convince yourself by substituting actual numbers. Hence, the elasticity reflects the fact that if rainfall changes by $\Delta\%$ then the quantity changes by $\alpha \Delta\%$.

**B. Logarithms and Growth Rates**

A mathematical function that is extremely useful in economics is that of a logarithm or log. The log of a number depends on the base, $b$ and is defined by the following:

$$y = \log_b x \text{ then } b^y = x.$$  

The most common base used in economics is the number $e = 2.718...$ This number is defined by the following series:

(B.1) $$e = \sum_{n=1}^{\infty} \frac{1}{n!}.$$  

This does not have much economic intuition, but another interpretation of $e$ is given by the following limit:

(B.2) $$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m.$$  

The economic interpretation is that $e$ represents the value of $1$ invested for one-year at 100% interest if compounded continuously. Why? Let $V(1)$ denote the value of $1$ after one year. In general,

(B.3) $$V(1) = V(1 + \text{interest}) = 1(1 + 1) = 2.$$  

If the interest is compounded semi-annually (every 6 months) then the investment yields 50% every 6 months. That is,

(B.4) $$V(1) = \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{2}\right).$$  

In general, if the interest is compounded $m$ times per year, then the value will be:

(B.5) $$V(1) = \left(1 + \frac{1}{m}\right)^m.$$  

Letting $m$ go to infinity yields the expression in eq. (B.2).

Logs are a useful concept because they convert (some) non-linear functions into linear functions. This is illustrated by the following set of rules (unless noted otherwise, all logs will be base $e$).

(I) If $y = x^z$, then $\log y = z \log x$.

(II) If $y = xz$, then $\log y = \log x + \log z$.

Note that (I) and (II) together imply that if
\[ y = \frac{x}{z} = xz^{-1} \Rightarrow \]
\[ \log y = \log x + \log(z^{-1}) = \log x - \log z \]

Logs (in particular, natural logs) are extremely useful when describing variables that are growing over time. This should not be surprising given the definition of \( e \) in (B.2) – the definition of \( e \) is inherently related to compounding. To examine this more closely, consider a variable \( x \) that is growing at the constant rate \( g \). Let the initial time period be period 0, then the value of \( x \) next period is:

\[ (B.6) \quad x_1 = x_0(1 + g) \]

Note that rearranging terms, this relationship implies that \( g \) is the percentage change in \( x \) from one period to the next:

\[ (B.7) \quad g = \frac{x_1 - x_0}{x_0} = \frac{\Delta x}{x_0} \]

Note that from eq. (B.6), the value of \( x \) in any period \( t \) can be written as:

\[ (B.8) \quad x_t = x_0(1 + g)^t \]

That is \( x \) is growing exponentially over time. Taking logs converts this non-linear relationship to a linear relationship:

\[ (B.9) \quad \log x_t = \log x_0 + t \log(1 + g) \]

But natural logs permit the following approximation if \( g \) is not too large:

\[ (B.10) \quad \log(1 + g) \approx g \]

So that

\[ (B.11) \quad \log x_t = \log x_0 + tg \]

Recall, that for all logs, the log of 1 is zero. (Why?) The approximation in eq. (B.10) implies that the natural log function has the slope of a 45 degree line. This can be seen in the graph below – for comparison 1 also graph log base 10.

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**Figure 2:** Plot of \( \log_{10}(x), \log_e(x) \)

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Note that the natural log function does indeed have a 45 degree slope in the neighborhood of unity.

This approximation is behind the “Rule of 72” which is used to calculate how long it will take for an investment to double in value. That is, in (B.11) we need to solve for \( t \) knowing that \( x_t = 2x_0 \). Making this substitution in (B.11) yields

\[ \log 2 + \log x_0 = \log x_0 + t \log(1 + g) \Rightarrow \]

\[ (B.12) \quad t = \frac{\log 2}{g} \]

The natural log of 2 is approximately 0.69 – not a convenient dividend. The number 0.72 is close enough and is easier to use. For comparison suppose $1000 is invested at a constant interest rate and left for 36 years. If the interest rate is 12\%, the rule of 72 implies that the quantity will double every 6 years. This means that after 36 years, the quantity will have grown to $64,000 \( \left(2^6 \times 1000 \right) \). If the yield is 6\%, the amount will double every 12 years growing to $8000 \( \left(2^3 \times 1000 \right) \).

Note that the final quantity from a 12\% yield is not simply the double of that implied by a
6% yield. This is the “miracle” of compound interest.

As mentioned in the previous section, percentages play a key role in much of our analysis. And logs facilitate this since the difference in logs of a variable determines its growth rate:

(B.13) \( \log x_t - \log x_0 = g \).

Keeping this in mind permits an easy characterization of the following. Suppose \( x \) and \( y \) are growing at the rate \( g_x \) and \( g_y \) respectively. Then if \( z = xy \), this implies:

(B.14) \( \log z_t = \log x_t + \log y_t \).

But then the growth rate of \( z \) is simply:

(B.15)
\[
g_z = \log z_{t+1} - \log z_t \\
= (\log x_{t+1} - \log x_t) + (\log y_{t+1} - \log y_t) \\
= g_x + g_y
\]

Likewise if \( z = \frac{x}{y} \), then

(B.16) \( g_z = g_x - g_y \).

Finally, if \( y \) is a constant and \( z = x^y \), then

(B.17) \( g_z = y g_x \).

A final important characteristic of logs is the derivative. Namely:

(B.18) \( \frac{d \log x}{dx} = \frac{1}{x} \).

Rewriting this as:

(B.19) \( d \log x = \frac{dx}{x} \).

shows that the derivative of a log of a variable is simply the percentage change of a variable.

This is a useful thing to remember. For example, at the end of Section 2.A we determined that the equilibrium price of oranges was:

(B.19) \( p = \frac{y}{\rho^a} \).

Taking logs and then taking derivatives yields:

(B.20) \( \frac{d \log p}{d \log \rho} = \eta = -\alpha \).

This is the same as the result in Section 2.A but obviously much easier. As you go on in Economics, you will see this transformation and interpretation repeatedly.