Bond Pricing Formulas

The basic bond pricing formula for payments made **annually** is:

$$P_b = \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \ldots + \frac{C_N}{(1+i)^N} = \sum_{t=1}^{N} \frac{C_t}{(1+i)^t}$$  \hspace{1cm} (1)$$

where: $C_t, \ t = (1, \ldots, N)$ denotes the cash payment received in period $t$ and $N$ denotes the maturity of the bond. For a coupon bond, $C_N$ will include both the coupon payment and the face value (denoted $F$) of the bond, while for an amortizing bond, $C_t = C$ for all $t$ – payments are constant. For a discount bond, $C_t = 0$ for all $t < N$ and $C_N = F$. Note that the **yield to maturity**, $i$, is expressed as an annual yield.

Now consider a coupon bond with payments that are made **semi-annually**. That is, the annual coupon payment of $C = rF$ (where $r$ is the **coupon rate**) is received twice a year. Then the bond pricing formula becomes:

$$P_b = \frac{C/2}{(1+i/2)} + \frac{C/2}{(1+i/2)^2} + \ldots + \frac{C/2}{(1+i/2)^{2N}} + \frac{F}{(1+i/2)^{2N}} = \sum_{t=1}^{2N} \frac{C/2}{(1+i/2)^t} + \frac{F}{(1+i/2)^{2N}}.$$  \hspace{1cm} (2)$$

Note that the implied **effective annual yield** is $(1 + i/2)^2$.

If payments are made at the rate of $m$ times per year (so that all but the last payment is $C/m$), the formula becomes:

$$P_b = \sum_{t=1}^{mN} \frac{C/m}{(1+i/m)^t} + \frac{F}{(1+i/m)^{mN}}.$$  \hspace{1cm} (3)$$

If $m = 2$, the formula (3) is the same as that in (2). Again the effective annual yield is $(1 + i/m)^m$.

**Continuous Compounding**  Suppose that the interest rate, $i$, is 100%. Then, using the above results, the future value, $FV$, of a dollar after one year which is compounded at the rate of $m$ times per year is:

$$FV = \left(1 + \frac{1}{m}\right)^m.$$  \hspace{1cm} (4)$$

If the rate becomes continuous, the future value is defined by the following limit:

$$FV = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = e = 2.71828..$$  \hspace{1cm} (5)$$

Hence continuous compounding of a 100% interest rate implies an effective annual yield of 171.828..%. If after 1 year a $1 becomes $e$, then after 2 years the dollar becomes $e^2$. Hence, $A$ dollars invested for $t$ years becomes $Ae^t$.

If $i \neq 1$, then the future value formula under non-continuous compounding is, in general,
\[ FV = \left(1 + \frac{i}{m}\right)^m = \left(\left(1 + \frac{i}{m}\right)^{\frac{m}{i}}\right)^i = \left(1 + \frac{1}{w}\right)^{wi} \quad \text{(where } w = \frac{m}{t}\text{).} \quad (6) \]

As before, continuous compounding is defined by the following limit:

\[ FV = \lim_{w \to \infty} \left[\left(1 + \frac{1}{w}\right)^{wi}\right] = e^i. \quad (7) \]

Using the same reasoning as before, an investment of \$A invested at rate \(i\) for \(t\) years will yield a future value of: \(FV = Ae^{it}\).

**Alternative representation of interest rates.**

The payment of \$1 after \(1/m\) of a year (so that a continual payment would be received \(m\) times per year) at annual rate of \(i\) generates a future value of \(\left(1 + \frac{i}{m}\right)\). Taking natural logs (and recalling that \(\ln (1 + x) \approx x\) if \(x\) is small), then the \(\ln FV = \frac{i}{m}\). Alternatively, we could think of an annual interest rate of \(i\) being invested for \(1/m\) of a year. Then the future value would be:

\[ FV = (1 + i)^{\frac{1}{m}}. \quad (8) \]

Again taking natural logs, \(\ln FV = \frac{1}{m} \ln (1 + i) \approx \frac{i}{m}\). Hence the two representations are *approximately* equivalent:

\[ FV = \left(1 + \frac{i}{m}\right)^{\frac{m}{i}} = (1 + i)^{\frac{1}{m}}. \quad (9) \]

To take an example, suppose you invest \$10,000 for \(1/4\) of a year at a 10% (annual) interest rate. Then, according to eq. (4), this amount would become:

\[ FV = 10,000 \left(1 + \frac{0.10}{4}\right) = 10,250. \]

Under the alternative characterization, the future value would be:

\[ FV = 10,000 \left(1 + 0.10\right)^{0.25} = 10,241. \]