Problem 1

Figures 1 and 2 below illustrate the most recent numbers from the CBO. The second row of Figure 2 illustrates annual GDP growth and highlights all US recessions as gray bars.
Figure 1: Tax Revenues and Defense Expenditures
Figure 2: Deficit, GDP growth, and Recessions

Total Deficit
Average = -2.3

[Graph showing the relationship between deficit, GDP growth, and recessions over time.]

Source: U.S. Department of Commerce, Bureau of Economic Analysis

Shaded areas indicate US recessions.
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Problem 2

In this problem we want to find the primary deficit, \( G_t - T_t \). To do so we use the government’s budget constraint

\[
P_t G_t + B_{t-1} R_{t-1} = P_t T_t + (B_t - B_{t-1}) + (M_t - M_{t-1}),
\]

where \( P_t \) is the price level at time \( t \), \( G_t \) is government spending (in real terms) at time \( t \), \( B_t \) represent the dollar amount in outstanding government bonds at time \( t \), \( T_t \) denotes the tax revenue (in real terms) at time \( t \). Some simple manipulations allow us to rewrite equation (1) in terms of the primary deficit:

\[
P_t (G_t - T_t) = \frac{B_t + M_t}{P_t} - \frac{M_{t-1}}{P_t} \left( 1 + R_{t-1} \right) \frac{B_{t-1}}{P_t} - \frac{1}{1 + \pi_{t-1}} \left( 1 + R_{t-1} \right) \frac{B_{t-1}}{P_t},
\]

where we define the current rate of inflation, i.e. from period \( t-1 \) to period \( t \), as \( \pi_{t-1} \equiv \frac{P_t}{P_{t-1}} - 1 \). Since we know that for any point in time, \( t = 0, 1, 2, 3, ... \), \( \frac{M_t}{P_t} = m = 200 \) and \( \frac{B_t}{P_t} = b = 400 \) are both constant we can collect terms and write the primary deficit as

\[
(G_t - T_t) = \left[ 1 - \frac{1}{1 + \pi_{t-1}} \right] \cdot m + \left[ 1 - \frac{1 + R_{t-1}}{1 + \pi_{t-1}} \right] \cdot b.
\]

Finally, using \( R_{t-1} = 0.06 \) and \( \pi_{t-1} = 0.05 \) we find that the primary deficit amounts to

\[
(G_t - T_t) = \left[ 1 - \frac{1}{1.05} \right] \cdot 200 + \left[ 1 - \frac{1.06}{1.05} \right] \cdot 400
\]

\[
= 0.0476 \cdot 200 - 0.0095 \cdot 400
\]

\[
= 9.5238 - 3.8095 = 5.7143
\]

units of aggregate production for every point in time, \( t = 0, 1, 2, ... \). Notice that the real value of debt and outstanding real money balances are substantial (multiple hundred billion units of aggregate production) even though the primary deficit is tiny (only about 6 billion units of aggregate production).
Problem 3

We derived in class that if a country’s GDP growth, \( n \equiv \frac{Y_t}{Y_{t-1}} - 1 \), and the real interest rate, \( r \), are constant then it’s debt to GDP ratio, \( b_t \equiv \frac{B_t}{Y_t} \), evolves according to the difference equation

\[
b_t = d_t + \frac{1 + r}{1 + n} \cdot b_{t-1}, \tag{5}\]

where \( d_t \equiv \frac{D_t}{Y_t} \) expresses the country’s deficit as a fraction of GDP. Given that \( d_t = d = 0.08 \) for all time periods, \( t = 0, 1, 2, ..., r = 0.05 \), and \( n = 0.06 \), we can easily solve for the country’s steady state debt to GDP ratio, \( b \):

\[
\begin{align*}
    b & = d + \frac{1 + r}{1 + n} \cdot b \\
    \left[1 - \frac{1 + r}{1 + n}\right] \cdot b & = d \\
    b & = \left[\frac{1}{1 - \frac{1 + r}{1 + n}}\right] \cdot d \\
    & = \left[\frac{1 + n}{n - r}\right] \cdot d = \left[\frac{1.06}{0.06 - 0.05}\right] \cdot 0.08 \\
    & = \left[\frac{1.06}{0.01}\right] \cdot 0.08 = 106 \cdot 0.08 \\
    & = 8.48 \approx 848\% \tag{6}
\end{align*}
\]

Problem 4

In this problem we relax one of the many strong assumptions necessary in order to establish Ricardian equivalence: We assume that households have to pay a higher interest rate on loans than the government does. This has two implications: (1) the problem becomes significantly more complicated, and, more importantly, (2) the timing of lump-sum taxes starts to matter for households’ decisions, even if the government does not change its path of expenditure.

Part (a) If the household does not borrow at all we say it stays in autarky and its path of consumption is as follows

\[
c_{1}^{\text{aut}} = y_1 - T_1 = 1 - T_1
\]
\[ c_2^{\text{aut}} = y_2 - T_2 = (1 + r) - T_2 \]  \tag{7}

Plotting this particular choice of consumption in the \((c_1, c_2)\)-space gives the household’s \textit{no lend/no borrow} point. For the rest of this problem it is important to realize that the \textit{no lend/no borrow} point is \textit{always} feasible and, therefore, will \textit{always} be on the household’s budget line (whatever the budget line may look like).

For all scenarios below, the easiest way to construct the household’s budget set (all the feasible combinations of \(c_1\) and \(c_2\)) is to follow a 4-step procedure:

1. Record the \textit{no lend/no borrow} point in the \((c_1, c_2)\)-space.
2. Ask yourself what would happen if the household decides to consume \textit{only} in period 2? In other words, what is \(c_2\) if \(c_1 = 0\)? In this situation the household saves \textit{all its wealth} in order to consume as much as possible in period 2. Record the point \((0, c_2^{\text{max}})\) (Hint: this point must be on the vertical axis).
3. Ask yourself what would happen if the household decides to consume \textit{only} in period 1? In other words, what is \(c_1\) if \(c_2 = 0\)? In this situation the household borrows \textit{as much as possible} in order to consume as much as possible in period 1 and nothing in period 2. Record the point \((c_1^{\text{max}}, 0)\) (Hint: this point must be on the horizontal axis).
4. Connect the 3 points that you just found. The part of the budget line between the point on the vertical axis and the \textit{no lend/no borrow} point are the household’s options to save in order to shift consumption toward period 2. The part between then \textit{no lend/no borrow} point and the horizontal axis represents the household’s options to borrow and shift consumption toward period 1.

\textbf{Part (b)} Following the procedure described above we find the following:

1. We already did this in part (a).
2. The maximum amount that the household can consume in period 2 amounts to \(c_2^{\text{max}} = 1 + r\). Notice that this is completely independent
of the government’s choice of inter-temporal tax scheme.

\[ c_1 = 0 \implies S = y_1 - T_1 = 1 - T_1 \]
\[ c_2^{\text{max}} = y_2 - T_2 + (1 + r) \cdot S \]
\[ = (1 + r) - T_2 + (1 + r) \cdot (1 - T_1) \]
\[ = 2 \cdot (1 + r) - [(1 + r) \cdot T_1 + T_2] \]
\[ = 2 \cdot (1 + r) - (1 + r) \cdot \overline{G} \]
\[ = 1 + r, \quad (8) \]

where \( S \) denotes the household’s savings in period 1. Notice that we made use of the fact that (no matter what \( T_1 \) and \( T_2 \) are) the government always needs to ensure that \( T_1 + \frac{T_2}{1+r} = \overline{G} = 1 \) and hence

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\[(1 + r) \cdot T_1 + T_2 \] = \((1 + r) \cdot G = (1 + r).\]

(3) For the sake of exposition we will first ask ourselves how much the household could consume in period 1 if it had the possibility to borrow at rate \(r\) (which it can’t). It turns out that this amount is \(c_{1}^{\text{max},r} = 1).\)

This is because of the following

\[c_2 = 0 \implies B = c_1^{\text{max},r} - (y_1 - T_1) = c_1^{\text{max},r} - 1 + T_1\]

\[c_2 = 0 = y_2 - T_2 - (1 + r) \cdot B\]

\[= (1 + r) - T_2 - (1 + r) \cdot (c_1^{\text{max},r} - 1 + T_1)\]

where \(B\) denotes the amount of borrowing to ensure maximum consumption in period 1. Solving for \(c_1^{\text{max},r}\) then yields

\[(1 + r)c_1^{\text{max},r} = 2 \cdot (1 + r) - [(1 + r) \cdot T_1 + T_2]\]

\[= 2 \cdot (1 + r) - (1 + r) \cdot G = 1 + r\]

\[c_1^{\text{max},r} = 1.\] (10)

Notice, that this point is also independent of the particular choice of \(T_1\) and \(T_2\). This is because everything we did so far was using the interest rate \(r\), which is the government’s rate for saving and borrowing. Unfortunately, the household has to pay a premium if it wants to borrow and it can only borrow at the rate \(r^b > b\). Therefore, the maximum amount that the household can really consume in period 1, \(c_1^{\text{max}}\) is derived as follows:

\[c_2 = 0 = y_2 - T_2 - (1 + r^b) \cdot B\]

\[= (1 + r) - T_2 - (1 + r^b) \cdot (c_1^{\text{max}} - 1 + T_1)\]

\[(1 + r^b)c_1^{\text{max}} = (1 + r) - (1 + r^b) - \left[T_2 + (1 + r^b)T_1\right]_{>T_2+(1+r)T_1} \]

\[c_1^{\text{max}} = \frac{1 + r}{1 + r^b} + 1 - \frac{T_2}{1 + r^b} - T_1\]

\[= \frac{1 + r}{1 + r^b} + 1 - \frac{(1 + r) \cdot [1 - T_1]}{1 + r^b} - T_1\]

\[= \frac{1 + r}{1 + r^b} + 1 - \frac{1 + r}{1 + r^b} + \frac{(1 + r)}{1 + r^b} \cdot T_1 - T_1\]

\[= 1 - \left[1 - \frac{1 + r}{1 + r^b}\right] \cdot T_1 < 1,\] (11)
where we made use of the fact that the government budget constraint implies that $T_2 = (1 + r) \cdot [1 - T_1]$. This has two important implications: First, unsurprisingly, agents are not able to borrow just as much as they would if they could borrow at the same interest rate as the government. Second, the amount of consumption in period 1, $c_1^{\text{max}}$, now depends on the particular choice of the tax path $(T_1, T_2)$. The higher $T_1$ the less they can consume in period 1. This is because the government invests the tax revenue, $T_1$, at a lower return than the households have to pay for their loan. Hence the implicit “tax refund” in period 2, $(1 + r) T_1$, is less than the equivalent loan repayment, $(1 + r_b) T_1$, if the household wanted to offset the tax burden in period 1.

(4) Connecting the 3 points we just found illustrates that the higher borrowing rate, $r^b$, induces a kink in the household’s budget constraint and hence makes the budget set smaller. Figure 3 illustrates this. In the region where the household saves (to the left of the autarky point) the budget line has a slope of $-(1 + r)$ while in the region to the right (where the household borrows) the slope is $-(1 + r^b)$ which is steeper. This means the agents have fewer choices available and depending on their preferences this might indeed affect their choices. If the household was very impatient and wanted to consume a lot in period 1, then its choices would definitely be affected by the government’s choice of tax policy.

Part (c) Now assume that the government chose the tax policy $T_1 = 0$ and $T_2 = (1 + r) \cdot \overline{G} = 1 + r$. With no borrowing and lending $c_1 = 1$ and $c_2 = (1 + r) - T_1 = 0$. This means the endowment point is already on the horizontal axis which implies that the household does not have to borrow at all in order to achieve the maximum possible consumption in period 1, i.e. $c_1 = c_1^{\text{max}}$. This means that the only relevant interest rate for the household is the lending rate, $r$, and the household has the exact same budget set as a household who could borrow and lend at the interest rate $r$. The entire budget line now has the slope $-(1 + r)$. From equation (11) in part (b) we can see that this is the only tax scheme where the household does not need the borrowing rate $r^b$. Figure 4 shows the graphical representation.

Part (d) If the tax policy is $T_1 = \overline{G} = 1$ and $T_2 = 0$ the autarky point is given by $c_1 = 0$ and $c_2 = 1 + r$. This point is on the vertical axis and hence
Figure 4: Budget Set with $T_1 = 0$ and $T_2 = 1 + r$

$c_2 = c_2^{\text{max}} = 1 + r$. This means that the government taxes the household so heavily that it cannot afford to consume at all in period one. Hence, the only way for the household to leave this point is to borrow at the expensive borrowing rate, $r_b$. Using equation (11) we see that $c_1^{\text{max}} = \frac{1}{1+r} < 1$. Now, the entire budget line has the slope $-(1 + r^b)$. This is illustrated in Figure 5.

**Part (e)** A benevolent government would choose the scenario from part (c) (taxing only in period 2). This gives the household the largest possible choice set (the area including the budget line and and below). Basically, by postponing its tax collection, the government allows households to take
advantage of the saving rate $r$ and not have to borrow at all at $r^b$. The other way to think of this is that the government is borrowing on behalf of the household at the low rate of $r$ ($T_1 = 0$ suggests a deficit if $G_1 > 0$).

**Problem 5**

Here the government runs a deficit of 1 at time 0 and a “core balanced budget” in all other periods. This means that the government never repays the interest cost of the initial debt nor the principle. As a result, the government will have to continually roll over its debt each period (by issuing new debt). The government has not backed its initial borrowing with any future revenues
so it does not intend to repay its debt. This will violate the transversality condition.

More precisely, let \( B_t \) represent the stock of debt in period \( t \). Then

\[
B_0 = G_0 - T_0 = 1 \\
B_1 = (1 + r) + (G_1 - T_1) = 1 + r \\
B_2 = (1 + r)^2 \\
\vdots \\
B_t = (1 + r)^t
\]

and substituting into the transversality condition gives

\[
\lim_{t \to \infty} \frac{1}{(1 + r)^t} B_t = \lim_{t \to \infty} \frac{1}{(1 + r)^t} (1 + r)^t = 1 \neq 0. \quad (12)
\]

### Problem 6

The budget constraints in each period are

\[
Y = c_0 + B_0 \\
Y + B_0 (1 + r) = c_1 + B_1 \\
B_1 (1 + r) = c_2
\]

Solve these for the bonds to yield the intertemporal budget constraint:

\[
Y + \frac{Y}{1 + r} = c_0 + \frac{c_1}{1 + r} + \frac{c_2}{(1 + r)^2}
\]

b. The Lagrangian is:

\[
L = \ln c_0 + \beta \ln c_1 + \beta^2 \ln c_2 + \lambda \left( Y + \frac{Y}{1 + r} - c_0 - \frac{c_1}{1 + r} - \frac{c_2}{(1 + r)^2} \right)
\]

The necessary conditions are:

\[
\frac{\partial L}{\partial c_0} = 0 \implies \frac{1}{c_0} = \lambda \\
\frac{\partial L}{\partial c_1} = 0 \implies \beta \frac{1}{c_1} = \lambda \left( \frac{1}{1 + r} \right) \\
\frac{\partial L}{\partial c_2} = 0 \implies \beta^2 \frac{1}{c_2} = \lambda \left( \frac{1}{1 + r} \right)^2
\]
These represent the MC = MB condition that always characterize a maximum in economics.

c. If $\beta = (1 + r)^{-1}$, then the necessary conditions imply consumption is constant: $c_0 = c_1 = c_2 = \bar{c}$. From the budget constraint we have

$$\bar{c} = \frac{Y \left(1 + \frac{1}{1+r}\right)}{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2}}$$

d. If $\beta > (1 + r)^{-1}$, then note from the necessary conditions we have

$$\frac{c_1}{c_0} = \frac{\beta}{1 + r} > 1$$

$$\frac{c_2}{c_1} = \left(\frac{\beta}{1 + r}\right)^2 > 1$$

These imply $c_0 < c_1 < c_2$. The reason is that agents are less impatient (in terms of their discounting future utility) than implied by the current interest rate.