1 Answers to Final

1. This question had several elements (and was recycled from the final in 2000):

1. Habit persistence implies lagged consumption is a state variable.

2. The fact that the endowment is growing means that equity prices will be a function of both \((x_t, \lambda_t)\). The assumption of CRRA preferences implies that, even in the presence of habit persistence (as modeled in this question), the function will be homogeneous of degree 1 in \(x_t\). (Same as in Mehra-Prescott.)

3. The definition of equilibrium is critical - refer to Lucas (1978). It is defined in terms of two functions: a value function and a price function (for equity).

Here is the dynamic programming problem:

\[
v(c_{t-1}, z_{t-1}, x_t, \lambda_t) = \max \left\{ \frac{(c_t - \alpha c_{t-1})^{1-\gamma}}{1-\gamma} + \beta E \left[ v(c_t, z_t, x_{t+1}, \lambda_{t+1}) \right] \right\}
\]

\[
+ \mu_t [z_{t-1} (q_t + x_t) - c_t - q_t z_t]
\]

Using the envelope theorem produces:

\[
q_t \mu_t = \beta E \left[ \mu_{t+1} (q_{t+1} + x_{t+1}) \right]
\]

\[
\mu_t = (c_t - \alpha c_{t-1})^{-\gamma} - \alpha \beta E \left[ (c_{t+1} - \alpha c_t)^{-\gamma} \right]
\]

Equilibrium is defined by a value function and a price function as described in Lucas (1978) - refer to this and study it carefully. This is very important since this concept of equilibrium is to a large extent the basis for modern macroeconomics.

To solve for equilibrium, we need to be guided by the fact that this is a growing economy. Hence equity price will be growing and, hence, will not be stationary. We use the insight from Mehra and Prescott’s analysis in a similar economy to conjecture that the price function will be homogeneous of degree 1 in the endowment - the price-dividend ratio will be function of the growth rate only. Since the price function must satisfy eq.(2), it is critical that the \(\mu_t\) cooperate. The form of habit persistence and preferences ensures this since the Lagrange multiplier can be written as (imposing equilibrium that \(c_t = x_t\) and \(z_t = 1\)):

\[
\mu_t = x_t^{-\gamma} \left[ \lambda_t^\gamma (\lambda_t - \alpha)^{-\gamma} - K \right]
\]
where

\[ K = \alpha \beta E \left[ (\lambda_{t+1} - \alpha)^{-\gamma} \right] \]

is a constant due to the i.i.d. assumption. We now use this in eq. (2) - along with the conjecture that the price function is homogeneous of degree 1 in output, i.e. \( q(x_t, \lambda_t) = x_t \omega(\lambda_t) \). If the conjecture is correct, then the level of the endowment will drop out of the equation and only the growth rate terms will remain. This indeed is the case as eq. (2) becomes:

\[ \left[ \lambda_t^\gamma (\lambda_t - \alpha)^{-\gamma} - K \right] \omega(\lambda_t) = \beta E \left\{ \lambda_{t+1}^{1-\gamma} \left[ \lambda_{t+1}^\gamma (\lambda_{t+1} - \alpha)^{-\gamma} - K \right] \right\} \] (5)

Note that the RHS is a constant again due to the i.i.d. assumption. This expression implies that the price-dividend ratio is an increasing function of the growth rate, i.e. \( \omega(\lambda_t) > 0 \). Also note that in the absence of habit persistence \( (\alpha = 0) \), \( \omega(\lambda_t) = \bar{\omega} \) a constant. So habit persistence produces volatility in this ratio.

2. For this question I was looking for a discussion that included the following points. First, that the impulse mechanisms in RBC models have evolved over time: from technology shocks, to fiscal policy shocks (govt. expenditures and taxes (possibly distortionary)), to shocks to household production, to taste shocks - or some combination of these. The amplification mechanisms primarily focus on the labor market with intertemporal substitution of leisure being the main feature. Household production models would also include labor shifting in and out of market activities; models with variable capacity utilization (or capital utilization) would have this additional feature. Propagation mechanisms include investment and the law of motion of capital in the basic model. Recent models have included adjustment costs to labor and habit persistence in preferences could play a role as well. The evolution of the mechanisms included in the framework reflect the two areas that the model has failed noticeably: labor market characteristics (too little labor volatility and too high a correlation between productivity and labor) and output dynamics (basically there are no effective internal propagation mechanisms in the basic model - the Cogley and Nason observation). Also, as many of you pointed out, the amplification and propagation mechanisms in the basic RBC model depend critically on the relevant substitution effects outweighing the income effects - but with persistence technology shocks, the opposite occurs. All the ideas neatly packaged constitutes a good and complete answer.

3. Most everyone got the first two parts correct (although a few of you put money in the utility function ????), the final part proved to be the most difficult. I go straight to the first order conditions after applying the envelope theorem (and letting \( \lambda_t \) and \( \gamma_t \) represent the Lagrange multipliers on the budget and CIA constraints respectively):

\[ U_t' = \lambda_t + \gamma_t \] (6)
\[ \lambda_t = \beta \left[ \lambda_{t+1} (f'_{t+1} + 1 - \delta) \right] \] (7)
\[ \lambda_t \frac{1}{P_t} = (1 + n_t) \beta \left[ \lambda_{t+1} \frac{1}{P_{t+1}} \right] \] (8)
\[ \lambda_t \frac{1}{P_t} = \beta U'_{t+1} \frac{1}{P_{t+1}} \] (9)

The first condition demonstrates that, because of the CIA constraint, the shadow price of real wealth does not equal the marginal utility of consumption if the CIA constraint is binding - i.e. the form of wealth matters. Equation (7) is the standard intertemporal efficiency condition for capital, the next equation is the analogous expression for nominal bonds. The last expression implies that the utility cost of a dollar acquired in period \( t \) is equal to the discounted utility of consumption that dollar will provide in the following period.

b. From eq.(7) we have immediately that in steady state (where \( \lambda_t = \lambda \), \( f'(K_{t+1}) = f'(K) \)), money growth will not affect the steady-state capital stock.

c. As the problem states and was demonstrated in class, in a MIUF model (like Sidrauski’s) we have the following expression:

\[ \frac{U_{m,t}}{U_{c,t}} = \frac{n_t}{1 + n_t} \]

That is, the marginal rate of substitution between real balances and consumption is equal to the present discounted value of the current nominal interest rate.

In a CIA model, the Lagrange multiplier on the CIA constraint is the equivalent to the marginal utility of real balances - use this insight to guide the analysis.

First, note that eq. (8) can be written as:

\[ \frac{1}{1 + n_t} = \frac{\beta \lambda_{t+1}/P_{t+1}}{\lambda_t/P_t} \] (10)

while, using eq.(6), eq. (9) can be written as:

\[ 1 - \frac{\beta \lambda_{t+1}/P_{t+1}}{\lambda_t/P_t} = \frac{\beta \gamma_{t+1}/P_{t+1}}{\lambda_t/P_t} \] (11)

so that combining eq.(11) with eq.(10) yields:

\[ \frac{\beta \gamma_{t+1}/P_{t+1}}{\lambda_t/P_t} = \frac{n_t}{1 + n_t} \] (12)

This makes sense in this model - the relevant shadow price of a dollar chosen in period \( t \) is next period’s Lagrange multiplier on the CIA constraint. Again using eq.(6)

\[ \frac{\gamma_{t+1}}{U'_{t+1}} = \frac{n_t}{1 + n_t} \]

This demonstrates that nominal interest rates are zero if and only if the CIA constraint (next period) is not binding.
4. (This question was also recycled from the Final in Spring 2000). I go straight to the equilibrium necessary conditions:

\[ c_t = \frac{1 - \alpha}{A} k_t^\alpha h_t^{-\alpha} \]  

(13)

\[ c_t^{-1} = \beta \left\{ c_{t+1}^{-1} \left[ \alpha k_{t+1}^\alpha h_{t+1}^{1-\alpha} + 1 - \delta \right] \right\} \]  

(14)

\[ k_{t+1} = k_t (1 - \delta) + x_t \]  

(15)

\[ k_t^\alpha h_t^{1-\alpha} = y_t = c_t + x_t \]  

(16)

The steady-state of this economy is characterized by constant growth in consumption, output, and capital - but growth is balanced so that the capital output ratio is constant. Labor is also constant. We are given that:

\[ \beta = 0.99; \quad \frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t} = \nu = 1.05, \]  

(17)

\[ \frac{x_t}{k_t} = \bar{x} = 0.07, \quad \bar{h} = 0.25, \quad k_t \quad \frac{\bar{k}}{y_t} = 4 \]  

(18)

From eq.(15) we derive (and using the given relationships):

\[ \frac{k_{t+1}}{k_t} = 1 - \delta + \frac{x_t}{k_t} = 1 \Rightarrow \delta = 0.02 \]  

(19)

From the intertemporal efficiency condition we obtain:

\[ \frac{c_{t+1}}{c_t} = \beta \left[ \alpha \bar{y} k + 1 - \bar{h} \right] \Rightarrow \alpha = 0.322 \]  

(20)

From the intratemporal efficiency condition we obtain:

\[ \bar{h} = \frac{1 - \alpha}{A} \frac{\bar{y}}{\bar{c}} \]  

(21)

Note that the law of motion for the capital stock can be written as:

\[ \frac{k_{t+1}}{y_{t+1}} \frac{y_{t+1}}{y_t} = \frac{k_t}{y_t} (1 - \delta) + 1 - \frac{c_t}{y_t} \Rightarrow \frac{\bar{c}}{y} = 0.72 \]

Using this in eq.(21) yields \( A = 3.76 \).

Finally, we have

\[ \nu = \frac{y_{t+1}}{y_t} = \frac{z_{t+1} k_t^\alpha h_t^{1-\alpha}}{z_t k_t^\alpha h_t^{1-\alpha}} = \theta \left( \frac{k_{t+1}}{k_t} \right)^\alpha = \theta \nu^\alpha \Rightarrow \theta = 1.033 \]