1. a. The dynamic programming problem associated with the question is:

\[
V(z_{t-1}, b_{t-1}, f_{t-2}; x_t) = \max_{(c_t, z_t, b_t, f_t)} \left\{ U(c_t) + \beta E V(z_t, b_t, f_t, f_{t-1}; x_{t+1}) \right\} + \lambda_t \left[ z_{t-1}(q_t + x_t) + b_{t-1} + f_{t-2} - c_t - p_t b_t - \phi_{t-1} f_{t-1} \right]
\]

The arguments of the value function denote the state variables. Note that prices have been suppressed (for simplicity) since, in equilibrium, these are a function of the aggregate state variable, \( x_t \). Also note that, since the endowment is distributed as i.i.d., there is no time subscript on the expectations operator. The first order conditions associated with bonds and equity have been derived numerous times, so I will not do that here. Focusing on the forward contract, we have the derivative:

\[
\beta E \left[ \frac{\partial V(z_t, b_t, f_t, f_{t-1}; x_{t+1})}{\partial f_t} \right] = 0 \quad (2)
\]

This derivative is the change in utility due to forward contracts purchased in the previous period. Applying the envelope theorem to eq. (1) yields:

\[
\frac{\partial V(z_{t-1}, b_{t-1}, f_{t-2}; x_t)}{\partial f_{t-1}} = \beta E \left[ \frac{\partial V(z_t, b_t, f_t, f_{t-1}; x_{t+1})}{\partial f_{t-1}} \right] - \lambda_t \phi_{t-1} \quad (3)
\]

Note that the derivative on the RHS of the above expression represents the change in utility due to forward contracts purchased 2 periods ago. We need to apply the envelope theorem to that by taking the following derivative

\[
\frac{\partial V(z_{t-1}, b_{t-1}, f_{t-2}; x_t)}{\partial f_{t-2}} = \lambda_t \quad (4)
\]

Updating eq. (4) and using this in eq. (3) and then updating that expression to use in eq. (2) we have:

\[
\beta E \left\{ \beta E [U'_{t+2}] - U'_{t+1} \phi_t \right\} = 0
\]

(where we have used the condition that \( \lambda_t = U'_t \)). Or, using the law of iterated expectations:

\[
\beta E [U'_{t+2}] = E[U'_{t+1}] \phi_t \quad (5)
\]

Eq. (5) has the standard MC=MB interpretation: the RHS represents the expected utility loss from buying the contract at time \( t \) while the LHS represents
the expected utility gain from the return in period \( t + 2 \). At an optimum, these must be equal. The necessary conditions associated with equity and bonds are:

\[
q_t = \frac{\beta E \left[ U'_{t+1} \left( q_{t+1} + x_{t+1} \right) \right]}{U'_t}
\]

\[
p_t = \frac{\beta E \left[ U'_{t+1} \right]}{U'_t}
\]

c. A recursive competitive equilibrium is defined by four functions: the functions defining the prices of the three assets, \( q(x), p(x), \phi(x) \), and a value function \( V(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t) \), such that, (i) given \( q(x), p(x), \phi(x) \), \( V(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t) \) solves the consumer’s maximization problem and (ii) markets clear. The rational expectations assumption in this context is that the price functions agents use to solve their maximization problem are the same implied by market clearing.

d. The assumption that the endowment is i.i.d. implies the numerators in the prices for equity and bonds are constant. Hence, this establishes that both prices are increasing in the endowment. The price of the forward contract, in contrast, will be constant since it is determined by the ratio of two forecasts both of which are invariant over time.

2. The definition of equilibrium need not be as complete (note that the consistency condition between individual and aggregate policy functions is the assumption of rational expectations) but it should have the functions noted and make a distinction between the individual and aggregate state variables. Also, the social planner problem should be stated - the equilibrium behavior should be described. Here are the details:

For the individual, the maximization problem is:

\[
v \left( z_{t-1}, \zeta_t \right) = \max_{(c_t, h_t, z_t)} \left\{ \frac{c_t^{1-\gamma}}{1-\gamma} + A \left( 1 - h_t \right) + \beta E \left[ v \left( z_{t}, \zeta_{t+1} \right) \right] \right\}
\]

subject to:

\[
w_t h_t + z_{t-1} \left( \pi_t + q_t \right) = c_t + q_t z_t
\]

where \( z_t \) denotes equity holdings in period \( t \) and \( \pi_t = \zeta_t h_t^\alpha - w_t h_t \) denotes profits. Note the value function, \( v \left( z_{t-1}, \zeta_t \right) \), is a function of the individual’s beginning of period equity holdings and the current technology shock. Prices are also part of the state vector, but since these will be functions of the aggregate state \( (\zeta_t) \), I suppress these in the state vector.

The associated necessary conditions are (using \( \lambda_t \) to denote the Lagrange multiplier for the budget constraint):

\[
c_t : \quad c_t^{1-\gamma} - \lambda_t = 0
\]

\[
h_t : \quad -A + \lambda_t w_t = 0
\]

\[
z_t : \quad \beta E \left[ \frac{\partial V \left( z_t, \zeta_{t+1} \right)}{\partial z_t} \right] - \lambda_t q_t = 0
\]

Or, combining terms and using the envelope theorem:

\[
A c_t^\gamma = w_t
\]
\[
c_t^{-\gamma} q_t = \beta E \left[ c_{t+1}^{-\gamma} (\pi_{t+1} + q_{t+1}) \right]
\]  
(7)

These have the standard interpretations.

The firms maximization problem is static and is simply:

\[
\max_{h_t} (\zeta_t h_t^\alpha - w_t h_t)
\]

which leads to the necessary condition:

\[
\alpha \zeta_t h_t^{\alpha-1} = w_t
\]  
(8)

b. Definition of a recursive competitive equilibrium

A recursive competitive equilibrium can be defined by a set of functions: a value function that defines the household’s problem (6), a wage function, \( w(\zeta_t) \), an equity price function, \( q(\zeta_t) \), a set of decision rules for households, \( c(z_{t-1}, \zeta_t), h(z_{t-1}, \zeta_t) \), and a corresponding set of aggregate per capita decision rules, \( C(\zeta_t), H(\zeta_t) \). These functions must satisfy:

a. The household’s problem (as noted above for the value function.)

b. The necessary condition for households: i.e. \( q_t = q(\zeta_t) \) in eq. (7) when evaluated at market clearing quantities.

c. The necessary condition for firms: i.e. \( w_t = w(\zeta_t) \) in eq.(8).

d. The consistency of individual and aggregate decisions: \( c(1, \zeta_t) = C(\zeta_t), h(1, \zeta_t) = H(\zeta_t) \). (that is, market clearing requires \( z_t = 1 \forall t \).

e. The aggregate resource constraint: \( C(\zeta_t) = \zeta_t [H(\zeta_t)]^\alpha \)

c. Social planner problem

Since there is no capital, the social planner problem is simply to maximize utility each period given by:

\[
\max_{c_t, h_t} \left[ \frac{c_t^{1-\gamma}}{1-\gamma} + A (1 - h_t) \right]
\]

subject to \( c_t = \zeta_t h_t^\alpha \)

The necessary condition is:

\[
A c_t^{\gamma} = \alpha \zeta_t h_t^{\alpha-1}
\]  
(9)

This is clearly the same as in the competitive economy when the household’s and firm’s necessary conditions are combined. Note that equity has no relevance for the social planner problem since it does not influence equilibrium allocations.

d. Characterization of equilibrium

Using the necessary condition, eq.(9) and the resource constraint implies:

\[
h_t = \Gamma \zeta_t^{1-\alpha(1-\gamma)}
\]
where $\Gamma = \left(\frac{a}{A}\right)^{\frac{1}{1-\alpha-\gamma}}$. Given the restrictions on $(\alpha, \gamma)$ this establishes that labor is an increasing function of the technology shock. The wage rate is given by eq.(8) so that, in equilibrium:

$$w_t = \alpha \Gamma^{(\alpha-1)} \xi_t^{\frac{1}{1-\alpha-\gamma}}$$

so that wages are also increasing in the technology shock.

Equity prices are given by

$$q_t = \beta E_t \left[ c_t^{-\gamma} \left( \pi_t + q_{t+1} \right) \right]$$

In the numerator, all variables are functions of next period’s technology shock. Given the assumption of i.i.d. shocks, this implies the numerator is a constant. Since consumption is an increasing function of the technology shock, then equity prices will be as well.

3. For a change of pace, I solve this without making the substitution for $i_t$. The dynamic programming representation is:

$$V(k_t, z_t) = \max_{(c_t, i_t)} \{ \ln c_t + \beta E_t \left[ V(k_{t+1}, z_{t+1}) \right] \\ + \lambda_t \left[z_t \kappa_t^\alpha - c_t - i_t \right]$$

with the law of motion of capital given by

$$k_{t+1} = k_t^{1-\delta} \delta^{\delta-1}$$

The necessary conditions are:

$$\frac{1}{c_t} - \lambda_t = 0 \tag{10}$$

$$\beta E_t \left[ \frac{\partial V(\cdot_{t+1})}{\partial k_{t+1}} \left( \delta k_t^{1-\delta} \delta^{\delta-1} \right) \right] - \lambda_t = 0 \tag{11}$$

Using the envelope theorem:

$$\frac{\partial V(\cdot)}{\partial k_t} = \beta E_t \left[ \frac{\partial V(\cdot_{t+1})}{\partial k_{t+1}} \left( 1 - \delta \right) k_t^{-\delta} \delta^{\delta-1} + \lambda_t \alpha z_t \kappa_t^{\alpha-1} \right] \tag{12}$$

Note that eq.(11) implies

$$\beta E_t \left[ \frac{\partial V(\cdot_{t+1})}{\partial k_{t+1}} \right] = \frac{\lambda_t}{(\delta k_t^{1-\delta} \delta^{\delta-1})} \tag{13}$$

Using this in eq.(12) yields:

$$\frac{\partial V(\cdot)}{\partial k_t} = \lambda_t \left[ (1 - \delta) \frac{i_t}{k_t} + \alpha \frac{y_t}{k_t} \right] \tag{14}$$
Updating and using in eq.(11) results in (also using the fact that $(\delta k_t^{1-\delta} i_t^{-1}) = \delta k_{t+1} i_{t-1}^{-1}$)

$$
\beta E_t \left\{ \lambda_{t+1} \left[ \frac{(1 - \delta) i_{t+1}}{k_{t+1}} + \alpha \delta \frac{y_{t+1}}{c_{t+1}} \right] \right\} \left( \delta \frac{k_{t+1}}{i_t} \right) = \lambda_t \quad (15)
$$

Note that this intertemporal efficiency condition represents the relevant tradeoffs: The LHS is the additional output due to investment: the term in parentheses represents the additional capital produced through investment while the two terms in square brackets represent the depreciation term while the second is the MPK.

This can be written as:

$$
\beta E_t \left[ (1 - \delta) \frac{i_{t+1}}{c_{t+1}} + \alpha \delta \frac{y_{t+1}}{c_{t+1}} \right] = \frac{i_t}{c_t} \quad (16)
$$

Using the resource constraint yields:

$$
\zeta_0 + \zeta_1 E_t \left( \frac{y_{t+1}}{c_{t+1}} \right) = \frac{y_t}{c_t} \quad (17)
$$

Where

$$
\zeta_0 = 1 - \beta (1 - \delta) \\
\zeta_1 = \beta \left[ 1 - \delta (1 - \alpha) \right]
$$

Note that $\zeta_1 < 1$. Solving this expression through recursive substitution yields:

$$
\frac{y_t}{c_t} = \frac{\zeta_0}{1 - \zeta_1} = \frac{1 - \beta (1 - \delta)}{1 - \beta [1 - \delta (1 - \alpha)]} \quad (18)
$$

As a check, note that with 100% depreciation ($\delta = 1$), this simplifies to:

$$
c_t = (1 - \alpha \beta) y_t
$$

This is precisely the solution we obtain under the standard approach.

4. Letting the money transfer at time $t$ be denoted as $\mu \tilde{M}_{t-1}$, the associated dynamic programming problem is:

$$
V(M_{t-1}, B_{t-1}, k_t) = \max \left\{ U \left( c_t, \frac{M_{t-1}}{P_t} \right) + \beta V(M_t, B_t, k_{t+1}) \right\}
$$

$$
+ \lambda_t \left( w_t - c_t - k_{t+1} - \frac{M_t}{P_t} \right)
$$

with $w_t$ defined as real wealth, the law of motion is given by:

$$
w_{t+1} = f(k_{t+1}) + k_{t+1} (1 - \delta) + \frac{M_t}{P_{t+1}} + \frac{B_t}{P_{t+1}} (1 + n_t) + \frac{\mu \tilde{M}_t}{P_{t+1}}
$$
The associated first-order conditions for optimal \( M_t, k_{t+1}, B_t \) are respectively:

\[
U_{1,t} = \beta (U_{1,t+1} + U_{2,t+1}) \frac{P_t}{P_{t+1}} \tag{19}
\]

\[
U_{1,t} = \beta [U_{1,t+1} (f'(k_{t+1}) + 1 - \delta)] \tag{20}
\]

\[
U_{1,t} = \beta U_{1,t+1} (1 + n_t) \frac{P_t}{P_{t+1}} \tag{21}
\]

Note that, unlike the model studied in class, money chosen in period \( t \) affects the marginal utility of real balances in the following period (see eq. (19)).

A steady-state equilibrium is defined by the time-invariant 4-tuple \((c^*, k^*, m^*, n^*)\) that solve the necessary conditions and the resource constraint. The steady-state level of real balances is best defined in terms of the beginning of period level of the money stock since this enters into the utility function: In equilibrium \( M_{t-1} = \bar{M}_{t-1} \) and define \( m^* = \frac{M_{t-1}}{P_t} \). This implies that

\[
\frac{P_t}{P_{t+1}} = \frac{1}{1 + \mu} \nonumber
\]

In steady-state, the necessary conditions become:

\[
U_{1,t}^* = \beta (U_{1,t+1}^* + U_{2,t+1}^*) \frac{1}{(1 + \mu)} \tag{22}
\]

\[
1 = \beta (f'(k^*) + 1 - \delta) \tag{23}
\]

\[
1 = \beta (1 + n^*) \frac{1}{(1 + \mu)} \tag{24}
\]

From eq. (23) we have immediately that money growth has no effect on \( k^* \). Given the resource constraint, this implies that \( c^* \) is also independent of the monetary growth rate. The effect of money growth on real balances can be determined by taking the total differential of eq. (22). Making the assumption that preferences are separable in consumption and real balances (and that the utility function for real balances is strictly concave) directly proves that \( \frac{dm^*}{d\mu} < 0 \). The reason is that since money growth determines the inflation rate, increases in the money growth rate implies that real balances are a poor asset - so real balances fall in steady-state. From eq. (24), we see immediately that nominal interest rates are increasing in the money growth - this is due to the Fisher effect.