1. a. The dynamic programming problem associated with the question is:

\[
V(z_{t-1}, b_{t-1}, f_{t-2}; x_t) = \max_{(c_t, q_t, b_t, f_t)} \{ U(c_t) + \beta EV(z_t, b_t, f_t, f_{t-1}; x_{t+1}) \} + \lambda_t \left[ z_{t-1} (q_t + x_t) + b_{t-1} + f_{t-2} - c_t - p_t b_t - \phi_{t-1} f_{t-1} \right]
\]

The arguments of the value function denote the state variables. Note that prices have been suppressed (for simplicity) since, in equilibrium, these are a function of the aggregate state variable, \(x_t\). Also note that, since the endowment is distributed as i.i.d., there is no time subscript on the expectations operator. The first order conditions associated with bonds and equity have been derived numerous times, so I will not do that here. Focusing on the forward contract, we have the derivative:

\[
\beta E \left[ \frac{\partial V(z_t, b_t, f_t, f_{t-1}; x_{t+1})}{\partial f_t} \right] = 0
\]

This derivative is the change in utility due to forward contracts purchased in the previous period. Applying the envelope theorem to eq. (1) yields:

\[
\frac{\partial V(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t)}{\partial f_{t-1}} = \beta E \left[ \frac{\partial V(z_t, b_t, f_t, f_{t-1}; x_{t+1})}{\partial f_{t-1}} \right] - \lambda_t \phi_{t-1}
\]

Note that the derivative on the RHS of the above expression represents the change in utility due to forward contracts purchased 2 periods ago. We need to apply the envelope theorem to that by taking the following derivative

\[
\frac{\partial V(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t)}{\partial f_{t-2}} = \lambda_t
\]

Updating eq. (4) and using this in eq. (3) and then updating that expression to use in eq. (2) we have:

\[
\beta E \left\{ \beta E[U_{t+2}'] - U_{t+1}' \phi_t \right\} = 0
\]

(where we have used the condition that \(\lambda_t = U_t'\)). Or, using the law of iterated expectations:

\[
\beta E[U_{t+2}'] = E[U_{t+1}'] \phi_t
\]

Eq. (5) has the standard MC=MB interpretation: the RHS represents the expected utility loss from buying the contract at time \(t\) while the LHS represents the expected utility gain from the return in period \(t+2\). At an optimum, these must be equal. The necessary conditions associated with equity and bonds are:

\[
q_t = \frac{\beta E[U_{t+1}' (q_{t+1} + x_{t+1})]}{U_t'}
\]
\[ p_t = \frac{\beta E[U_{t+1}']}{{U'_t}} \]

c. Many answers confused the recursive structure of equity prices with the definition of a recursive equilibrium. Simply because the equity price expression can recursively be solved forward to yield the expression:

\[ q_t = \sum_{i=1}^{\infty} \beta E[U_{t+i}' (q_{t+i} + x_{t+i})] \]

has very little to do with the definition of equilibrium. A recursive competitive equilibrium is defined by four functions: the functions defining the prices of the three assets, \( q(x), p(x), \phi(x) \), and a value function \( V(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t) \), such that, (i) given \( q(x), p(x), \phi(x) \), \( V(z_{t-1}, b_{t-1}, f_{t-1}, f_{t-2}; x_t) \) solves the consumer’s maximization problem and (ii) markets clear. The rational expectations assumption in this context is that the price functions agents use to solve their maximization problem are the same implied by market clearing.

d. The assumption that the endowment is i.i.d. implies the numerators in the prices for equity and bonds are constant. Hence, this establishes that both prices are increasing in the endowment. The price of the forward contract, in contrast, will be constant since it is determined by the ratio of two forecasts both of which are invariant over time.

2. For a change of pace, I solve this without making the substitution for \( i_t \). The dynamic programming representation is:

\[ V(k_t, z_t) = \max_{(c_t, i_t)} \{ \ln c_t + \beta E_t [V(k_{t+1}, z_{t+1})] \}
+ \lambda_t [z_t k_t^{\alpha} - c_t - i_t] \]

with the law of motion of capital given by

\[ k_{t+1} = k_t^{1-\delta} \]

The necessary conditions are:

\[ \frac{1}{c_t} - \lambda_t = 0 \] (6)

\[ \beta E_t \left[ \frac{\partial V(t+1)}{\partial k_{t+1}} (\delta k_t^{1-\delta} z_t^{\delta}) \right] - \lambda_t = 0 \] (7)

Using the envelope theorem:

\[ \frac{\partial V(t)}{\partial k_t} = \beta E_t \left[ \frac{\partial V(t+1)}{\partial k_{t+1}} (1-\delta) k_t^{-\delta} z_t^{\delta} \right] + \lambda_t \alpha z_t k_t^{\alpha-1} \] (8)

Note that eq.(7) implies

\[ \beta E_t \left[ \frac{\partial V(t+1)}{\partial k_{t+1}} \right] = \frac{\lambda_t}{(\delta k_t^{1-\delta} z_t^{\delta})} \] (9)
Using this in eq.(8) yields:

\[
\frac{\partial V}{\partial \lambda_t} = \lambda_t \left[ \frac{(1 - \delta)}{\delta} \frac{i_t}{k_t} + \alpha \frac{y_t}{k_t} \right]
\]  

(10)

Updating and using in eq.(7) results in (also using the fact that \( \delta k_t^{1-\delta} k_t^{\delta-1} = \delta k_{t+1} k_t^{-1} \))

\[
\beta E_t \left\{ \lambda_{t+1} \left[ \frac{(1 - \delta)}{\delta} \frac{i_{t+1}}{k_{t+1}} + \alpha \frac{y_{t+1}}{k_{t+1}} \right] \right\} \left( \frac{k_{t+1}}{i_t} \right) = \lambda_t
\]  

(11)

Note that this intertemporal efficiency condition represents the relevant tradeoffs: The LHS is the additional output due to investment: the term in parentheses represents the additional capital produced through investment while the two terms in square brackets represent the depreciation term while the second is the MPK.

This can be written as:

\[
\beta E_t \left[ (1 - \delta) \frac{i_{t+1}}{c_{t+1}} + \alpha \frac{y_{t+1}}{c_{t+1}} \right] = \frac{i_t}{c_t}
\]  

(12)

Using the resource constraint yields:

\[
\zeta_0 + \zeta_1 E_t \left( \frac{y_{t+1}}{c_{t+1}} \right) = \frac{y_t}{c_t}
\]  

(13)

Where

\[
\zeta_0 = 1 - \beta (1 - \delta)
\]
\[
\zeta_1 = \beta [1 - \delta (1 - \alpha)]
\]

Note that \( \zeta_1 < 1 \). Solving this expression through recursive substitution yields:

\[
\frac{y_t}{c_t} = \frac{\zeta_0}{1 - \zeta_1} = \frac{1 - \beta (1 - \delta)}{1 - \beta [1 - \delta (1 - \alpha)]}
\]  

(14)

As a check, note that with 100\% depreciation (\( \delta = 1 \)), this simplifies to:

\[
c_t = (1 - \alpha \beta) y_t
\]

This is precisely the solution we obtain under the standard approach.

3. I was looking for something along the lines of the following. Assume a representative agent, exchange economy and assume that the endowment follows the following law of motion:

\[
x_{t+1} = \lambda_{t+1} x_t
\]

where \( \lambda_t \) is a random variable following a two-state Markov process. Possible realization of \( \lambda_t \) are

\[
\lambda_t = \begin{cases} 
\lambda_1 = \lambda - \delta \\
\lambda_2 = \lambda + \delta 
\end{cases}
\]

3
with the transition probability matrix assumed to be symmetric with diagonal elements of $\pi$. Agents preferences exhibit constant relative risk aversion denoted $\gamma$. Equilibrium one- and two-period interest rates will be functions of the endowment growth rate. Denoting these functions as $R_1(\lambda_i) = R_{1i}$, $R_2(\lambda_i) = R_{2i}$, they are defined by:

\[
R_{1i}^{-1} = \beta E_i (\lambda_{i+1}^{-\gamma}) \\
R_{2i}^{-2} = \beta^2 E_i (\lambda_{i+1}^{-\gamma} \lambda_{i+2}^{-\gamma})
\]

(Note: there are many ways to express the equilibrium interest rates, this is simply one of them.). The term premium conditional on the current state (i.e. the value of $\lambda_i$) is given by

\[
tp(\lambda_i) = tp_i = \frac{R_{2i}^2}{E_i (R_{1i+1})} - R_{1i}
\]

Finally, the unconditional term premium is given by (since the unconditional probabilities are $\left(\frac{1}{2}, \frac{1}{2}\right)$ given the symmetric transition probability matrix:

\[
E (tp) = \frac{tp_1 + tp_2}{2}
\]

Calibration of this model means determining values for preferences $(\beta, \gamma)$ and the endowment process $(\lambda, \delta, \pi)$. The endowment process parameters are chosen so that the moments of the Markov process match that of per-capita consumption growth. In particular: $E(\lambda_i) = \lambda$, $Var(\lambda_i) = \delta^2$, $Corr(\lambda_i, \lambda_{i-1}) = 2\pi - 1$. With these parameters determined, one can then solve the model’s implication for the unconditional term premium for reasonable values of the preference parameters. This can then be compared to the average term premium observed over the sample period.

4. See p. 27 in the article “The Equity Premium in Retrospect”.

4