Answers to midterm

1. The characterization of risk in the Lucas asset pricing model (also known more generally as the consumption based capital asset pricing model – CCAPM) is derived from agents’ desire to smooth their marginal utilities over time. The risk of an asset is determined by whether the asset’s return properties mitigate or exacerbate the volatility of the marginal utility of consumption \( u_t' \). Hence, whether the expected return on an asset is greater than that on a risk free asset depends entirely on the covariance between the asset’s return \( \rho_{t+1} \) and \( u_t' \). If this covariance is positive, then the asset pays out well (poorly) in those states of the world in which agents’ place high (low) value on consumption – the risk premium (defined to be the difference between the expected return on a risky asset and the return on a riskless asset) will be negative. A negative covariance implies a positive risk premium. This can be seen formally as follows. The return on a risky asset must satisfy:

\[
1 = \frac{\beta E_t[u_{t+1}'\rho_{t+1}]}{u_t'}.
\]

while the (gross) riskless rate is defined as:

\[
1 = r_t \frac{\beta E_t(u_{t+1}')}{u_t'}.
\]

Using the definition of covariance, the risk premium can be written as:

\[
E_t(\rho_{t+1}) - r_t = -\frac{r_t \beta \text{Cov}_t(u_{t+1}', \rho_{t+1})}{u_t'}.
\]

2. Hall’s test is based upon the necessary condition that arises from maximizing expected lifetime utility with time-separable preferences:

\[
u_t' = (1 + r)\beta E_t[u_{t+1}']
\]

The lesson is that dynamic optimization implies that the best forecast of next period’s marginal utility is today’s marginal utility. That is, in the attempt to smooth consumption over time, all relevant information will have been taken into account and used optimally. Hall made the further assumptions that utility is quadratic and the discount factor canceled with the interest rate. Then, the best forecast of next period’s consumption is today’s consumption – no other variables should have predictive power once current consumption is used. Hall found some support for this prediction.

3. Going directly to the necessary conditions:

\[
\text{equity: } q_t u_t' = \beta E[u_{t+1}'(q_{t+1} + x_{t+1})]
\]

\text{bonds: } u_t' = (1 + r)\beta E[u_{t+1}']

note that the assumption of i.i.d. growth rates of the endowment is reflected in the lack of a time subscript on the expectations term.

In equilibrium, the equity price will be a function of the state variables: \( (x_t, \lambda_t) \). This is potentially problematic since it implies the equity price, like the endowment itself, is growing over time – it is non-stationary. But the assumption of CRRA helps us out – it implies that the
equity price function will be homogeneous of degree one in the endowment: 

\[ q_t = q(x_t, \lambda_t) = x_t c(\lambda_t). \]

Using this in the necessary condition for equity results in:

\[ c(\lambda_t) = \beta E\left[ \lambda_t^{1-\gamma} (c(\lambda_{t+1}) + 1) \right]. \]

The right hand side involves only functions of next period’s growth rate hence it is a constant implying that the equilibrium price-dividend ratio will be as well. In equilibrium, we have,

\[ \bar{c} = \frac{\beta E(\lambda_{t+1}^{1-\gamma})}{1 - \beta E(\lambda_{t+1}^{1-\gamma})}. \]

Note that for \( \bar{c} > 0 \), this places restrictions on the parameters because of the term in the denominator. The price of equity is: \( q_t = \bar{c} x_t \). The equilibrium interest rate is a constant:

\[ (1 + r) = \frac{1}{\beta E(\lambda_{t+1}^{1-\gamma})}. \]

With log utility, the risk premium on equity is:

\[ ep_t = \beta^{-1} \left\{ E(\lambda) - \frac{1}{E(\lambda^{-1})} \right\}. \]

By Jensen’s inequality, the term in brackets is positive. This can also be seen by:

\[ \text{Cov}(\lambda, \lambda^{-1}) = E(\lambda \lambda^{-1}) - E(\lambda) E(\lambda^{-1}) = 1 - E(\lambda) E(\lambda^{-1}) < 0. \]

4. Again, working from the necessary conditions:

\( p_1, u'_t = \beta E[u'_{t+1}] \)

(2) two-period bonds

\[ p_2, u'_t = \beta E[u'_{t+1} p_{t+1}]. \]

Note that again the i.i.d. assumption is reflected in the lack of a time subscript on the expectations operator. Also, note that the expression for the price of one-period bonds can be updated one period and used (2):

\[ p_{2, t}, u'_t = \beta^2 E[u'_{t+2}]. \]

In equilibrium, the prices of both bonds will be functions of the endowment. Since the endowment is i.i.d., this implies that the expectations terms in (1) and (2) are constant. Hence, bond prices are proportional. The volatility of bond prices is determined by the volatility of the marginal utility of consumption – hence, the elasticity of bond prices with respect to the endowment is equal to agents constant relative risk aversion parameter.